In The Name of God, The Merciful, The Beneficent

Stochastic Processes - CE695
Department of Computer Engineering
Sharif University of Technology
Midterm Exam - Fall 2005
(Due Saturday 19^{th} of Azar)

1. **[20 points] Basic Concepts**
   a. (10 points) Suppose you have a real random variable, X, that is Gaussian with zero mean and a variance of σ^2. Now take two independent, identically distributed Bernoulli random variables, Y_1 and Y_2 with
   $$p(Y_i = 1) = 0.5,\ p(Y_i = -1) = 0.5$$
   Define two new random variables: $W_1 = Y_1 X$ and $W_2 = Y_2 X$.
1. Find the probability density functions of W_1 and W_2.
2. Find the $E[W_1 W_2]$.
3. Are W_1 and W_2 uncorrelated?
4. Find the probability $W_1 = 0$ given $W_2 = 1$. Find the probability $W_1 = 1$ given $W_2 = 1$. Are the two random variables W_1 and W_2 independent? Why?

   b. (5 points) Show that if **x**(t) is a stochastic process with zero mean and autocorrelation $f(t_1)f(t_2)w(t_1-t_2)$, then the process **y**(t) = **x**(t)/f(t) is WSS with autocorrelation $w(\tau)$. If **x**(t) is white noise with autocorrelation $q(t_1)\sigma(t_1-t_2)$, then the process $z(t) = x(t)/\sqrt{q(t)}$ is WSS white noise with autocorrelation σ (t).

   c. (5 points) Show that if in an LTI system the output-input relation is given by **y**(t) = **x**(t+a) – **x**(t-a), then
   $R_y(\tau) = 2R_x(\tau) - R_x(\tau + 2a) - R_x(\tau - 2a)$ and $S_y(\omega) = 4S_x(\omega)\sin^2 a\omega$.

2. **[20 points] Linear Systems & Stochastic Processes**
   a. (10 points) Suppose $x_1(t)$ and $x_2(t)$, $-\infty < t < \infty$ are two independent wide-sense stationary processes with correlation function $R_{x1}(\tau)$ and $R_{x2}(\tau)$, respectively. Determine the condition for stationarity of the processes $y(t) = a\, x_1(t) - b\, x_2(t)$ and $z(t) = x_1(t)\, x_2(t)$. Find their autocorrelation functions. Is the independence assumption necessary? (a and b are constants).

   b. (10 points) The process x(t) is normal with zero mean, autocorrelation $R_x(\tau)$ and PSD $S_x(\omega)$. Show that the PSD of the process $y(t) = x^2(t)$ is given by
   $S_y(\omega) = 2\pi R^2(0)\sigma(\omega) + 2S_x(\omega)^* S_x(\omega)$

3. [25 points] Misc.

Consider an experiment in which a point with coordinates (ω_1 , ω_2) is drawn at random from the unit square:

$\Omega = \{(\omega_1, \omega_2) : 0 \le \omega_1, \omega_2 \le 1\}$.

A continuous-Parameter random process (field) is defined on the same square according to

$\underline{f}_{\omega_1,\omega_2}(x, y) = \text{sgn}[(x - \omega_1)(y - \omega_2)],\ 0 \le x\ ,\ y \le 1$.

a) Draw a typical sample function of this process.

b) What is its probability?

c) Calculate E$\{\ \underline{f}(x, y)\ \}$.

d) Calculate the second moment E$\{\ \underline{f}^2(x, y)\ \}$. and the variance $\sigma^2_{\underline{f}}(x, y)$ of this random field.

e) Calculate the autocorrelation $R_{\underline{ff}}(x_1, y_1, x_2, y_2) = E\{\ \underline{f}(x_1, y_1)\underline{f}(x_2, y_2)\ \}$

4. [20 points] Misc.

A 1-D w.s.s AR Process $\underline{f}(m)$, m=0,1,2,… is defined as follows:

$\underline{f}(0)$ has zero mean and variance σ_0^2

$\underline{f}(m+1) = \alpha \underline{f}(m) + \underline{w}(m),\quad m \ge 0$

Where α is a constant. And $\underline{w}(m)$ is an i.i.d. sequence with zero mean and variance σ_w^2, which is also independent of $\underline{f}(0)$

a) find E$\{\ \underline{f}(m)\ \}$.

b) find $R_{\underline{ff}}(m, n) = E\{\ \underline{f}(m)\underline{f}(n)\ \}$

5. [20 points] Misc.

Prove the "sufficient" Part of the theorem concerning the discrete Karhunen-Loeve transform:

Let $\underline{\beta}_{k\ell}\, k = 0,...,M-1, \quad \ell = 0,...,N-1$ be the basis images associated with the M×N unitary transform

$\underline{F} = \underline{P}\, \underline{f}\, \underline{Q}$,

And assume that $\underline{f}(m,n)$ is a zero mean random field with autocorrelation function

$R_{\underline{ff}}(m,n;r,s) = E\{\ \underline{f}(m,n)\underline{f}(r,s)\ \}$

Show that if the basis images satisfy the equation

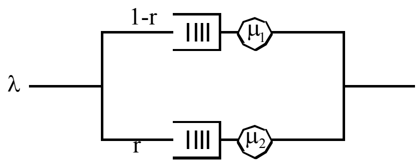
$$\sum_{r=0}^{M-1}\sum_{s=0}^{N-1} R_{\underline{ff}}(m,n;r,s)\left[\underline{\beta}_{k\ell}\right]_{rs} = \gamma_{k\ell}\left[\underline{\beta}_{k\ell}\right]_{mn}$$

For a set of constants $\gamma_{k\ell}$, then the transform coefficients are uncorrelated, i.e.

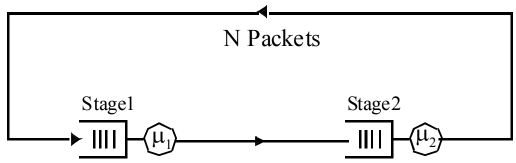
$\mathrm{E}\{\ \underline{F}(k,\ell)\underline{\mathrm{F}}(\mathrm{r,s})\ \} = \sigma^2_{\underline{F}(k,\ell)}\,\delta(k-r,\ell-s)$.

6. [25 points] Markov Chains and Queueing Theory

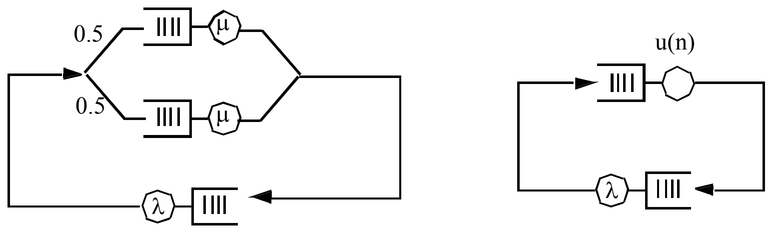
a. Consider the following queueing system with Poisson arrivals and two exponential servers with the given service and branch rates. Find *r* such that the time delay of the queue is minimized.



b. Consider the following cyclic queue. A total of *N* packets circulate around through two queueing facilities as shown. Both servers are of exponential type with the specified rates. Let P_k denote the probability that *k* packets are in stage 1 and *N*-*k* packets are in stage 2. Draw the Markov chain state transition diagram, and write down the relationship among P_k's for $k = 0,..., N$.



c. Consier the following sliding window flow control model and its equivalent



d. Find the rate u(n) of the equivalent network.

e. Let p(n) be the equilibrium probability that there are n packets in the upper queue in the equivalent network. Write down the balance equation for p(n).