Ergodicity

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Abstract

This module introduce ergodicity, such as Mean Ergodic, Correlation Ergodic.

Ergodicity

Many stationary random processes are also **Ergodic**. For an Ergodic Random Process we can exchange **Ensemble Averages** for **Time Averages**. This is equivalent to assuming that our ensemble of random signals is just composed of all possible time shifts of a single signal $X(t)$.

Recall from our previous discussion of Expectation[1] that the expectation of a function of a random variable is given by

$$E[g(X)] = \int g(x) f_X(x)\, dx \tag{1}$$

This result also applies if we have a **random function** $g(.)$ of a **deterministic variable** such as t. Hence

$$E[g(t)] = \int g(t) f_T(t)\, dt \tag{2}$$

Because t is linearly increasing, the pdf $f_T(t)$ is uniform over our measurement interval, say $-T$ to T, and will be $\frac{1}{2T}$ to make the pdf valid (integral = 1). Hence

$$\begin{aligned} E[g(t)] &= \int_{-T}^{T} g(t) \frac{1}{2T} dt \\ &= \frac{1}{2T} \int_{-T}^{T} g(t)\, dt \end{aligned} \tag{3}$$

If we wish to measure over all time, then we take the limit as $T \to \infty$.

This leads to the following results for Ergodic WSS random processes:

- **Mean Ergodic:**

$$\begin{aligned} E[X(t)] &= \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx \\ &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)\, dt \end{aligned} \tag{4}$$

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[1] http://cnx.rice.edu/content/m11068/latest/#eq4

- **Correlation Ergodic:**

$$\begin{aligned} r_{XX}(\tau) &= E\left[X(t)\,X(t+\tau)\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t),X(t+\tau)}(x_1,x_2)\,dx_1 dx_2 \\ &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,X(t+\tau)\,dt \end{aligned} \quad (5)$$

  and similarly for other correlation or covariance functions.

Ergodicity greatly simplifies the measurement of WSS processes and it is often assumed when estimating moments (or correlations) for such processes.

In almost all practical situations, processes are stationary *only over some limited time interval* (say T_1 to T_2) rather than over all time. In that case we deliberately keep the limits of the integral finite and adjust $f_{X(t)}$ accordingly. For example the autocorrelation function is then measured using

$$r_{XX}(\tau) = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} X(t)\,X(t+\tau)\,dt \quad (6)$$

This avoids including samples of X which have incorrect statistics, but it can suffer from errors due to limited sample size.