1. Solve the following problems from your text book (by Papoulis) chapter 15:
   15-1
   15-2
   15-8
   15-10
   15-12
   15-13

Solve these questions:

1) Consider a production line where each manufactured item may be defective with probability \( p \in (0, 1) \). The following inspection plan is proposed with a view to detecting defective items without checking every single one.
   It has 2 phases: In phase A, the probability of inspecting an article is \( r \in (0, 1) \). In phase B, all the articles are inspected. One switches from phase A to phase B as soon as a defective item is detected. One switches from phase B to phase A as soon as a sequence of \( N \) successive acceptable items has been found.
   Find the long-run proportion of items inspected. (Hint: Try to take states carefully such that long-run proportion of visiting a set of states equals to the long-run proportion of items inspected).

2) Consider a Markov process with the transition probability diagram shown below.

   ![Transition Diagram]

   a. Under what conditions on the \( a \)'s is the process recurrent?
   b. Under what conditions on the \( a \)'s is the process positive recurrent?
   c. Find the equilibrium distribution \( \pi \) in terms of \( a \)'s.

3)
a. Consider the Markov chain with transition matrix \( P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \) and initial distribution \( \pi_0 = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \). For each \( n \), \( Y_n = \begin{cases} 0, & \text{if } X_n = 1 \\ 1, & \text{otherwise} \end{cases} \) Show that \( (Y_0, Y_1, \ldots) \) is \textit{not} a Markov chain.

b. Let \( (X_0, X_1, \ldots) \) be a Markov chain with transition matrix \( P \). Define \( (Y_0, \ldots) \) by defining \( Y_n = X_{2n} \). Is \( (Y_0, \ldots) \) a Markov chain? If so, find its transition matrix (in terms of \( P \)).

4) Consider the following HMM:

![HMM Diagram]

\[
\begin{array}{c}
\begin{array}{c}
\text{Start Here with Prob. 1} \\
\end{array}
\end{array}
\]

\[
\begin{array}{cccccccc}
\alpha_{11} &=& 1/2 & \alpha_{12} &=& 1/2 & \alpha_{13} &=& 0 \\
\alpha_{21} &=& 0 & \alpha_{22} &=& 1/2 & \alpha_{23} &=& 1/2 \\
\alpha_{31} &=& 0 & \alpha_{32} &=& 0 & \alpha_{33} &=& 1 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
b_1(X) &=& 1/2 & b_1(Y) &=& 1/2 & b_1(\bar{Z}) &=& 0 \\
b_2(X) &=& 1/2 & b_2(Y) &=& 0 & b_2(\bar{Z}) &=& 1/2 \\
b_3(X) &=& 0 & b_3(Y) &=& 1/2 & b_3(\bar{Z}) &=& 1/2 \\
\end{array}
\]

\[
\begin{array}{cccc}
\pi_1 &=& 1 & \pi_2 &=& 0 & \pi_3 &=& 0 \\
\end{array}
\]

Where

\[
\alpha_i = P(q_{t+i} = S_i | q_t = S_j) \\
b_i(k) = P(O_i = k | q_t = S_j)
\]

Suppose we have observed this sequence: XZXYZZYZZ

(In long-hand: \( O_1 = X, O_2 = Z, O_3 = X, O_4 = Y, O_5 = Y, O_6 = Z, O_7 = Y, O_8 = Z, O_9 = Z \).)

Fill in the following table with \( \alpha_i(t) \) values, remembering the definition:

\[
\alpha_i(t) = P(O_1 \wedge O_2 \wedge \ldots \wedge O_t \wedge q_i = S_j)
\]

So for example, \( \alpha_1(2) = P(O_1 = X \wedge O_2 = Z \wedge O_3 = X \wedge q_1 = S_2) \).

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Warning: this is a question that will take a few minutes if you really understand HMMs, but could take hours if you don't!