Homework 4 (Chapter 4)

Problems

1. Compute the Fourier transform of each of the following signals: (P 4.21 (a, c, f, g, i) p. 338 and 2 extra parts.)
   a. \( x(t) = e^{-|t|} \cos 2t \)
   b. \([e^{-\alpha t} \cos \omega_0 t]u(t), \alpha > 0\)
   c. \( x(t) = (1 - |t|)u(t+1)u(1-t) \)
   d. \( x(t) = \left( \frac{\sin \pi t}{\pi(t-\frac{1}{2})} \right)^2 \)
   e. \( x(t) \) as shown in Figure P4.21(a) in p. 338 of textbook.
   f. \( x(t) = \begin{cases} 1 + t^2, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases} \)
   g. The signal \( x(t) \) depicted below:

2. Determine the continuous-time signal corresponding to each of the following transforms:
   a. \( X(j\omega) = 5[\delta(\omega + 1) - \delta(\omega - 1)] - 2j[\delta(\omega - \pi) + \delta(\omega + \pi)] \)
   b. \( X(j\omega) = 2 \cos(3\omega - \pi/3) \)

3. Determine which, if any, of the real signals depicted in below have Fourier transforms that satisfy each of the following conditions:
   1. \( \text{Re}\{X(j\omega)\} = 0 \)
   2. \( \text{Im}\{X(j\omega)\} = 0 \)
   3. There exists a real \( \alpha \) such that \( e^{j\alpha \omega} X(j\omega) \) is real
   4. \( \int_{-\infty}^{\infty} X(j\omega)d\omega = 0 \)
   5. \( \int_{-\infty}^{\infty} \omega X(j\omega)d\omega = 0 \)
   6. \( X(j\omega) \) is periodic
4. Let $X(j\omega)$ denote the Fourier Transform of the signal $x(t)$ depicted below.

   a. Find $X(j\omega)$.
   b. Find $X(0)$.
   c. Find $\int_{-\infty}^{\infty} X(j\omega) \, d\omega$.
   d. Evaluate $\int_{-\infty}^{\infty} X(j\omega) \frac{2\sin \omega}{\omega} e^{j2\omega} \, d\omega$. 

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e. Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.

f. Sketch the inverse Fourier transform of $\mathcal{R}e\{X(j\omega)\}$.

**Note:** You should perform all these calculations without explicitly evaluating $X(j\omega)$.

5. Find the impulse response of a system with the following frequency response: (P 4.18 p. 337)

$$H(j\omega) = \frac{(\sin^2(3\omega)) \cos \omega}{\omega^2}.$$ 

6. Consider a causal, discrete-time LTI system $F : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]$ whose input and output signals $x$ and $y$, respectively, satisfy the following linear, constant-coefficient difference equation:

$$y(n) + \alpha^2 y(n - 2) = x(n) + x(n - 2)$$

where $0 < \alpha < 1$.

a. Determine an expression for the frequency response $F : \mathbb{R} \to \mathbb{C}$ of the system.

b. Suppose $\alpha = 0.95$. Using a geometric (graphical) analysis, provide a well-labeled sketch of $|F(j\omega)|$, the magnitude of the frequency response of the system. Explain why this filter is called a notch filter.

c. Suppose $\alpha = 0.95$ and that the input signal is characterized by

$$\forall n \in \mathbb{Z}, x(n) = \cos(\frac{\pi}{2}n) + \frac{1}{3} \sin(\frac{\pi}{4}n) + 3 + (-1)^n.$$ 

Using little to no mathematical manipulation, determine a reasonable approximation for the corresponding output signal values $y(n)$. What assumption did you have to make about the phase response $\varphi$$F$ that enabled you to approximate the output signal $y$? Why was your assumption about the phase reasonable?

7. (Orthogonality-Preserving Property of the CTFT) In this problem, we set out to prove that the continuous-time Fourier transform (CTFT) preserves mutual orthogonality of signals, and that the inverse of the CTFT preserves mutual orthogonality of signal spectra. Consider a set $\phi_k, k \in \mathbb{Z}$, of mutually-orthogonal functions

$$\phi_k : \mathbb{R} \to \mathbb{C}$$

each of whose elements $\phi_k$ has finite energy $E_\phi$, i.e.

$$\langle \phi_k, \phi_l \rangle \overset{\Delta}{=} \int_{-\infty}^{\infty} \phi_k(t)\phi_l^*(t)dt = E_\phi \delta(k - l).$$

where $\delta$ is the Kronecker delta function and $^*$ denotes complex conjugation. Let $\hat{\phi}_k$ be the CTFT (spectrum) of $\phi_k$, i.e., for $k \in \mathbb{Z},$

$$\hat{\phi}_k : \mathbb{R} \to \mathbb{C}$$

$$\forall \omega \in \mathbb{R}, \hat{\phi}_k(\omega) = \int_{-\infty}^{\infty} \phi_k(t)e^{-j\omega t}dt$$

Show that

$$\langle \hat{\phi}_k, \hat{\phi}_l \rangle \overset{\Delta}{=} \int_{-\infty}^{\infty} \hat{\phi}_k(\omega)\hat{\phi}_l^*(\omega)d\omega = 2\pi E_\phi \delta(k - l).$$
Practical Assignment

1. Consider a discrete-time system $H_1$ with impulse response

$$h_1[n] = \delta[n] + \delta[n - 1] - \delta[n - 2] - \delta[n - 3],$$

a discrete-time system $H_2$ with impulse response

$$h_2[n] = \left(\frac{1}{2}\right)^n (u[n + 3] - u[n - 3]),$$

and a discrete-time signal

$$x[n] = \left(\frac{1}{4}\right)^n (u[n] - u[n - 6]).$$

The signals $h_1[n]$, $h_2[n]$, and $x[n]$ are all defined for $-8 \leq n \leq 8$.

a. Plot $h_1[n]$, $h_2[n]$, and $x[n]$ together using the `subplot` function.

b. Consider a system $H$ formed from the series connection of $H_1$ and $H_2$, where $x[n]$ is input to $H_1$, the output $v[n]$ of $H_1$ is input to $H_2$, and the output of $H_2$ is $y[n]$. Use the `conv` function to find $v[n]$ and $y[n]$. Plot $v[n]$ and $y[n]$ using the `subplot` function.

c. Now assume that the order of the systems is reversed, so that $x[n]$ is input to $H_2$, the output $v[n]$ of $H_2$ is input to $H_1$, and $y[n]$ is the output of $H_1$. Plot $v[n]$ and $y[n]$. Briefly explain why $v[n]$ is different in parts (b) and (c), whereas $y[n]$ is the same in both parts.