Stochastic Processes

Markov Chains
Absorption (cont'd)

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Absorbing Markov Chain

- An absorbing state is one in which the probability that the process remains in that state once it enters the state is 1 (i.e., $p_{ii} = 1$).
- A Markov chain is *absorbing* if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).
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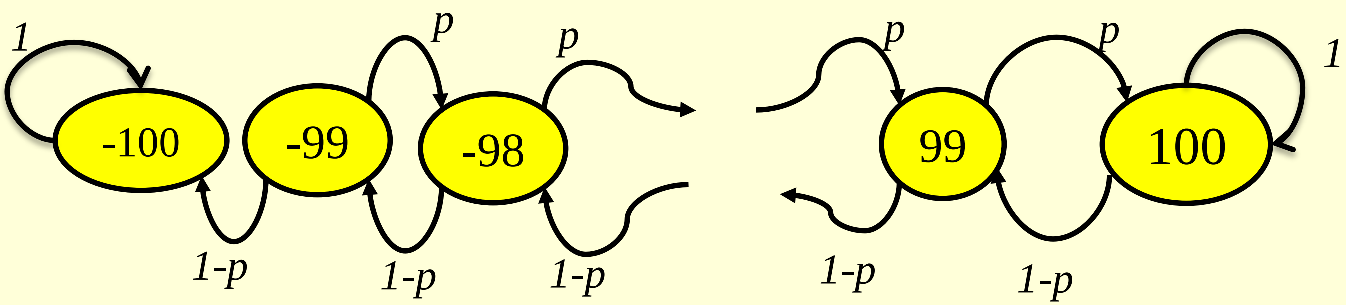
- Example: A 1D random walk starts from 0 and ends at -100 or 100. The states 100 and -100 are absorbing.



The canonical form

The transition matrix of any absorbing markov chain can be written as:

$$\mathbf{P} = \begin{array}{c} \\ \text{TR.} \\ \text{ABS.} \end{array} \begin{array}{c} \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \left(\begin{array}{c|c} \mathbf{Q} & \mathbf{R} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) \end{array}$$

The (i,j)'th entry of P^n is p_{ij}^n. However:

$$\mathbf{P}^n = \begin{array}{c} \\ \text{TR.} \\ \text{ABS.} \end{array} \begin{array}{c} \begin{array}{cc} \text{TR.} & \text{ABS.} \end{array} \\ \left(\begin{array}{c|c} \mathbf{Q}^n & * \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right) \end{array}$$

Absorption theorem

- In an absorbing Markov chain the probability that the process will be absorbed is 1. (i.e. $Q^n \to 0$ as $n \to \infty$).
- Proof: From each non-absorbing state s_j it is possible to reach an absorbing state starting from s_j. Therefore there exists p and m, such that the probability of not absorbing after m steps is at most p, in 2n steps at most p^2, etc. Since the probability of not being absorbed is monotonically decreasing, hence $\lim_{n\to\infty} Q^n = 0$.

The Fundamental Matrix

Definition: For an absorbing Markov chain P, the matrix $N = (I - Q)^{-1}$ is called the fundamental matrix for P.

Theorem: For and absorbing Markov chain

- the matrix $I - Q$ has an inverse N,
- and $N = I + Q + Q^2 + \cdots$.
- The ij-entry n_{ij} of the Matrix N is the expected number of times the chain is in state s_j, given that it starts in state s_i.

Proof:

- $(I - Q)x = 0 \Rightarrow x = Qx \Rightarrow x = Q^n x.$

  Since $\mathrm{Q^n} \to 0$, we have $Q^n x \to 0$, so $x = 0$.

  Thus $x = 0$ is the only point in the null-space of $I - Q$, therefore $(I - Q)^{-1} = N$ exists.

- $(I - Q)(I + Q + Q^2 + \cdots + Q^n) = I - Q^{n+1} \Rightarrow$

  $I + Q + Q^2 + \cdots + Q^n = N(I - Q^{n+1}).$

  Letting n tend to infinity we have:

$$N = I + Q + Q^2 + \cdots$$

Proof (cont'd):

- Suppose fixed i and j, consider the initial state to be s_i.

  $X^{(k)}$: a R.V. which equals 1 if the chain is in state s_j after k steps, and equals 0 otherwise.

  We have: $P\left(X^{(k)} = 1\right) = q_{ij}^{(k)}$

  The expected number of times the chain is in state s_j in the first n steps, given that it starts in state s_i is:

  $$E\left(X^{(0)} + X^{(1)} + \cdots + X^{(n)}\right) = q_{ij}^{(0)} + q_{ij}^{(1)} + \cdots + q_{ij}^{(n)}$$

  Letting n tend to infinity we have:

  $$E\left(X^{(0)} + X^{(1)} + \cdots\right) = q_{ij}^{(0)} + q_{ij}^{(1)} + \cdots = n_{ij}$$

Example:

- In the 1D Random walk example (between 0 and 4), the transition matrix in canonical form is:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 0 & 4 \end{array} \\ \left(\begin{array}{ccc|cc} 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} .$$

$$\mathbf{Q} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} ,$$

$$\mathbf{I} - \mathbf{Q} = \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}$$

$$\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \end{array}$$

- If we start in state 2, then the expected number of times in states 1, 2 and 3 before being absorbed are 1, 2 and 1.

Time to Absorption:

- **Question:** Given that the chain starts in state s_i, what is the expected number of steps before the chain is absorbed?

- **Reminder:** Starting from s_i, the expected number the process will be in state s_j before absorption is n_{ij}. Therefor $\sum_j n_{ij}$ is the expected number of steps before absorption.

- **Theorem:** Let t_i be the expected number of steps before the chain is absorbed, given that the chain starts in state s_i, and let t be the column vector whose i-th entry is t_i. Then $t = Nc$, where c is a column vector all of whose entries are 1.

Absorption Probabilities:

- **Question:** Given that the chain starts in the transient state s_i, what is the probability that it will be absorbed in the absorbing state s_j?

- **Intuition:** Starting from s_i, the expected number the process will be in state s_k before absorption is n_{ik}. Each time, the probability to move to state s_j is r_{kj} (kj-th element of matrix R introduced in the canonical form).

Absorption Probabilities:

- **Theorem:** Let b_{ij} be the probability that an absorbing chain will be absorbed in the absorbing state s_j if it starts in the transient state s_i. Let B be the matrix with entries b_{ij}. Then B is an t-by-r matrix, and $B = NR$, where N is the fundamental matrix and R is as in the canonical form.
- **Proof:**

$$\boldsymbol{B_{ij} = \sum_n \sum_k q_{ik}^{(n)} r_{kj}}$$

$$\boldsymbol{= \sum_k \sum_n q_{ik}^{(n)} r_{kj} = \sum_k n_{ik} r_{kj}}$$

$$\boldsymbol{= (NR)_{ij}}$$

References

- Grinstead C. M, and Snell J. L, Introduction to probability, American Mathematical Society, 1997