Homework 1 (Review Problems)

1. If \( A_1, A_2, \cdots, A_k \) are events with \( P(A_1 \cap A_2 \cap \cdots \cap A_{k-1}) > 0 \), prove that:
   \[
P(\bigcap_{i=1}^{k} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).
\]

2. Show that if the events \( A_1, A_2, \cdots, A_n \) are independent and \( B_i \) equals to \( A_i \) or \( A_i^c \) or \( S \), then the events \( B_1, B_2, \cdots, B_n \) are also independent.

3. Assume that \( A_1, A_2, A_3, A_4 \) are independent events and \( P(A_3 \cap A_4) > 0 \), show that:
   \[
P(A_1 \cup A_2 | A_3 \cap A_4) = P(A_1 \cup A_2)
\]

4. Assume that the random variable \( X \) has a probability mass function given by
   \[
P(x) = \begin{cases} 
a \times \left(\frac{1}{3}\right)^{x-2} & x = 2, 3, \ldots 
0 & \text{o.w.}
\end{cases}
\]
   (a) Find the value of \( a \) which makes \( P(x) \) a valid probability mass function.
   (b) Find the cumulative distribution function \( F_X(x) \).
   (c) Find \( P(4 \leq x < 6) \).
   (d) Find \( P(x > 3) \).

5. Suppose that \( X \) is a random variable with probability density function
   \[
f_X(x) = \begin{cases} 
\frac{2}{3} - a & 1 \leq x < 2 
-\frac{x}{2} + b & 2 \leq x \leq 4 
0 & \text{o.w.}
\end{cases}
\]
   and assume that \( E[X] = \frac{7}{5} \).
   (a) Find the values of \( a \) and \( b \) which make \( f_X(x) \) a valid probability density function.
   (b) Find \( F_X(x) \) and show that \( F_X(x) \) has the properties of a cumulative distribution function.
   (c) Find \( P(X = 3) \).
   (d) Find \( P(1.5 \leq X < 4) \).
   (e) Find \( P(1.5 \leq X < 4 | X \geq 3) \).
6. Suppose $X$ and $Y$ have joint density
\[ f_{XY}(x, y) = \begin{cases} \frac{1}{y}e^{-x/y}e^{-y} & x > 0, y > 0 \\ 0 & \text{o.w.} \end{cases} \]

Find $P(X > 1 | Y = y)$.

7. Suppose that $X$ and $Y$ are two random variables.
   (a) Prove that if $X$ and $Y$ are independent, they are also uncorrelated.
   (b) Give an example which $X$ and $Y$ are uncorrelated but not independent.
   (c) If $X$ and $Y$ are normally distributed and independent with the same variance, prove that $X - Y$ and $X + Y$ are independent.

8. Let $X$ represent a uniform random variable. Find $f_Y(y)$ if
   (a) $X \sim U(-2\pi, 2\pi)$ and $Y = X^3$.
   (b) $X \sim U(-2\pi, 2\pi)$ and $Y = X^4$.
   (c) $X \sim U(-\pi/2, \pi/2)$ and $Y = \tan(X)$.

9. Express the density $f_Y(y)$ of the random variable $Y = g(X)$ in terms of $f_X(x)$ if
   (a) $g(x) = |x|$.
   (b) $g(x) = e^{-x}U(x)$.

10. Assume that $X$ and $Y$ are two independent exponential random variables with parameters \( \lambda \) and \( \mu \) respectively.
   (a) Find the probability density function of $Z = \max(X, Y)$.
   (b) Find the probability density function of $Z = \min(X, Y)$.

11. Suppose that $X$ and $Y$ are two independent random variables. If $X \sim \text{uniform}(0, 2\pi)$ and $Y \sim \text{exponential}(1)$, Show that $W = \sqrt{2Y} \cos(X)$ and $Z = \sqrt{2Y} \sin(X)$ are independent random variables with standard normal distribution.

12. Show that if $X$ and $Y$ are two independent exponential random variables with $f_X(x) = e^{-x}U(x), f_Y(y) = e^{-y}U(y)$ and $Z = (X - Y)U(X - Y)$, then $E[Z] = \frac{1}{2}$.

13. For any two random variables $X$ and $Y$, prove that:
   (a) $E[E[X|Y]] = E[X]$.
   (b) $\text{var}(X) = E[\text{var}(X|Y)] + \text{var}(E[X|Y])$.

14. Consider two random variables $X$ and $Y$ with positive variances.
   (a) prove that $|\rho(X, Y)| \leq 1$ where $\rho(X, Y)$ is the correlation coefficient between $X$ and $Y$.
   (b) Show that if $Y - E[Y]$ is a positive or negative multiple of $X - E[X]$, then $\rho(X, Y) = 1$ or $\rho(X, Y) = -1$ respectively.
15. Assume that $X_1, X_2, \ldots$ are identically distributed random variables with expectation $\mu$ and variance $\sigma^2$. Let $N$ be a positive integer-valued random variable and $Y = X_1 + \ldots + X_N$.

(a) Show that $E[Y] = \mu E[N]$.
(b) Show that $\text{var}(Y) = \sigma^2 E[N] + \mu^2 \text{var}(N)$.

(The random variables $X_1, X_2, \ldots$ and $N$ are all independent.)

16. Assume that $X_1, X_2, \ldots$ are a sequence of independent and identically distributed continuous random variables. Consider another random variable $N$ such that:

$X_1 \geq X_2 \geq \cdots \geq X_{N-1} < X_N$

If $N \geq 2$, Show that $E[N] = e$. (Hint: $\sum_{i=0}^{\infty} \frac{1}{i!} = e$)

17. Let $x$ and $y$ be two randomly chosen natural numbers. Calculate the probability that these two numbers are relatively prime.