Homework 4 (Estimation Theory)

1. Which of the following statement are true, briefly explain your answer.

(a) MLE is function of every SS.
(b) MSS is function of every SS.
(c) MOM estimate is always unique.
(d) If \( X_i \overset{\text{iid}}{\sim} f(X_i|\theta) \) and \( T(X) \) be a SS for \( \theta \), then each statistic of \( X_i \)'s given \( T(X) \), i.e. \( E[R(X)|T(X)] \) is independent of \( \theta \).

2. Suppose \( X_1, X_2, \ldots, X_n \) are samples from following distributions. Find a Sufficient Statistic for each part.

(a) \( f(x|\rho) = \binom{k}{\rho} \rho^\rho (1-\rho)^{k-x} I_{\{0,1,\ldots,k\}}(x) \), with known \( k \).
(b) \( f(x|\theta_1, \theta_2) = U(\theta_1, \theta_2) \), with known \( \theta_2 \).
(c) \( f(x|\theta_1, \theta_2) = U(\theta_1, \theta_2) \)
(d) \( f(x|\theta) = U(-\theta, \theta) \)
(e) \( f(x|\theta) = \begin{cases} \frac{k \pi}{\theta^2} e^{-\frac{\theta^2}{x^2}} & x > 0 \\ 0 & \text{o.w.} \end{cases} \) \( k \) is constant and \( \theta \in (0, \infty) \).

3. Suppose \( X_1, X_2, \ldots, X_n \) are samples from following distributions. Find a Minimal Sufficient Statistic for each part.

(a) \( f(x|\theta) = U(-\theta, \theta) \)
(b) \( f(x|\theta) = N(x|\theta, \theta^2) \)
(c) \( f(x|\theta) = \frac{e^{-\frac{(x-\theta)^2}{2\theta^2}}}{(1+e^{-\frac{(x-\theta)^2}{\theta^2}})^2} \).
(d) \( f(x|\mu, \sigma) = \frac{1}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} I_{\{\mu, \infty\}}(x), \mu \in R \) and \( \sigma > 0 \) are both unknown.

4. Assume that we have \( X_1, \ldots, X_n \) from the probability density function:

\[
f_{X|\theta}(X|\theta) = \begin{cases} \frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}} & x > \theta \\ 0 & \text{o.w.} \end{cases}
\]

(a) Find a MSS for \( \theta \).
(b) Find a CSS for \( \theta \).

5. For each of the following pdfs let \( X_1, X_2, \ldots, X_n \) be iid observations. Find a complete sufficient statistic.

(a) \( f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}, -\infty \leq x \leq \infty, -\infty \leq \theta \leq \infty \).
(b) \( f(x|\mu, \sigma^2) = N(x|\mu, \sigma^2) \)
(c) \( f(x|\rho) = \binom{k}{x} \rho^x (1 - \rho)^{k-x} I_{\{0,1,\ldots,k\}}(x) \), with known \( k \).
(d) \( f(x|\theta) = e^{-(x-\theta)}, x \geq \theta. \)

6. Consider iid samples \( X_1, X_2, \ldots, X_n \) from an exponential family given by

\[
f(x|\theta) = h(x)c(\theta) \exp \left( \sum_{i=1}^{k} w_j(\theta) t_j(x) \right)\]

Show that

\[
T(X) = \left( \sum_{j=1}^{n} t_1(X_j), \ldots, \sum_{j=1}^{n} t_k(X_j) \right)
\]

(a) Is a sufficient statistic for \( \theta \).
(b) Is a minimal sufficient statistic for \( \theta \) if \( w_1(\theta), w_2(\theta), \ldots, w_k(\theta) \) be linear independent.