Signals & Systems

Chapter 7: Sampling

Adapted from: Lecture notes from MIT, Binghamton University, and Purdue

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Outline

1. The Concept and Representation of Periodic Sampling of a CT Signal

2. Analysis of Sampling in the Frequency Domain

3. The Sampling Theorem — the Nyquist Rate

4. In the Time Domain: Interpolation

5. Undersampling and Aliasing

6. Review/Examples of Sampling/Aliasing

7. DT Processing of CT Signals
SAMPLING

- most of the signals we encounter are CT signals, e.g. \( x(t) \).

**Question:** How do we convert them into DT signals \( x[n] \)?

- Sampling, taking snap shots of \( x(t) \) every \( T \) seconds.

- \( T \)-sampling period

\[ x[n] = x(nT), \quad n = ..., -1, 0, 1, 2, ... \] — regularly spaced samples

- Applications and Examples
  - Digital Processing of Signals
  - Images in Newspapers
  - Sampling Oscilloscope

How do we perform sampling?
Why/When Would a Set of Samples Be Adequate?

Observation: *Lots of signals have the same samples*

- By sampling we throw out lots of information.

**Key Question for Sampling:**

- Under what conditions can we reconstruct the original CT signal $x(t)$ from its samples?
Impulse Sampling—Multiplying $x(t)$ by the sampling function

\[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

\[ x_p(t) = x(t) \cdot p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) x(nT) \]
Analysis of Sampling in the Frequency Domain

\[ x_p(t) = x(t).p(t) \]

Multiplication Property \( \Rightarrow \)

\[ X_p(j\omega) = \frac{1}{2\pi} X(j\omega) \ast P(j\omega) \]

\[ P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \]

\[ \omega_s = \frac{2\pi}{T} = \text{Sampling Frequency} \]

Important to note:

\[ \omega_s \propto \frac{1}{T} \]

\[ X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j\omega) \ast \delta(\omega - k\omega_s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \]

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Illustration of sampling in the frequency-domain for a band-limited \( (X(j\omega) = 0 \text{ for } |\omega| > \omega_M) \) signal

\[
X_p(j\omega) \quad \text{drawn assuming} \quad \omega_s - \omega_M > \omega_M
\]

\[
i.e. \quad \omega_s > 2\omega_M
\]

No overlap between shifted spectra
Reconstruction of $x(t)$ from sampled signals

If there is no overlap between shifted spectra, a LPF can reproduce $x(t)$ from $x_p(t)$
Reconstruction (Continued)

Suppose $x(t)$ is bandlimited, so that

$$X(j\omega) = 0 \text{ for } |\omega| > \omega_M$$

Then $x(t)$ is uniquely determined by its samples $\{x(nT)\}$ if

$$\omega_s > 2\omega_M = \text{The Nyquist rate}$$

where $\omega_s = 2\pi/T$
Observations on Sampling

(1) In practice, we obviously don’t sample with impulses or implement ideal lowpass filters.
— One practical example: The Zero-Order Hold
Observations (Continued)

(2) Sampling is fundamentally a time varying operation, since we multiply x(t) with a time-varying function p(t). However,

\[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

is the identity system (which is TI) for bandlimited x(t) satisfying the sampling theorem \( \omega_s > 2\omega_M \).

(3) What if \( \omega_s \leq 2\omega_M \)? Something different: more later.
Time-Domain Interpretation of Reconstruction of Sampled Signals—Band-Limited Interpolation

\[ p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

\[ x(t) \rightarrow x_p(t) \rightarrow x_r(t) \]

\[ x_r(t) = x_p(t) * h(t), \quad \text{where } h(t) = \frac{T \sin \omega_c t}{\pi t} \]

\[ = ( \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)) * h(t) \]

\[ = \sum_{n=-\infty}^{\infty} x(nT)h(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin[\omega_c (t - nT)]}{\pi(t - nT)} \]

The lowpass filter interpolates the samples assuming \( x(t) \) contains no energy at frequencies \( \geq \omega_c \).
Graphic Illustration of Time-Domain Interpolation

Original CT signal

After Sampling

After passing the LPF
Interpolation Methods

- Bandlimited Interpolation
- Zero-Order Hold
- First-Order Hold — Linear interpolation

\[ p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \]

\[ x(t) \xrightarrow{\times} x_p(t) \xrightarrow{H(j\omega) h(t)} x_r(t) \]
Undersampling and Aliasing

When $\omega_s \leq 2\omega_M \Rightarrow \text{Undersampling}$
Undersampling and Aliasing (continued)

— Higher frequencies of $x(t)$ are “folded back” and take on the “aliases” of lower frequencies

— Note that at the sample times, $x_r(nT) = x(nT)$

$X_r(j\omega) \neq X(j\omega)$
Distortion because of \textit{aliasing}
Aliasing: What if the signal is NOT BANDLIMITED?

✧ For Non-BL Signals, Aliasing always happens regardless of $F_s$ value
✧ All practical signal are Non-BL!…so we choose $F_s$ to minimize aliasing.
✧ So we use a CT low-pass BEFORE the ADC! (demonstrated on next slide)
Aliasing: What if the signal is NOT BANDLIMTED?
(continued)
A Simple Example

\[ X(t) = \cos(w_0 t + \Phi) \]

Picture would be Modified…
Sampling Review

If \( X(j\omega) = 0, |\omega| > \omega_M \) and \( \omega_s = \frac{2\pi}{T} > 2\omega_M \)

then, assuming we choose \( \omega_M < \omega_c < \omega_s - \omega_M \):

\[ x_r(t) = x(t) \]
Strobe Demo

Δ > 0, strobed image moves forward, but at a slower pace

Δ = 0, strobed image still

Δ < 0, strobed image moves backward.

Applications of the strobe effect (aliasing can be useful sometimes):

E.g., Sampling oscilloscope

Demo: Effect of aliasing on music.
**DT Processing of Band-Limited CT Signals**

Why do this? **Inexpensive, versatile, and higher noise margin.**

How do we analyze this system?

—We will need to do it in the frequency domain in both CT and DT
—In order to avoid confusion about notations, specify

\( \omega \)—CT frequency variable
\( \Omega \)—DT frequency variable (\( \Omega = \omega T \))

Step 1: Find the relation between \( x_c(t) \) and \( x_d[n] \), or \( X_c(j\omega) \) and \( X_d(e^{j\Omega}) \)
Time-Domain Interpretation of C/D Conversion

Note: Not full analog/digital (A/D) conversion – not quantizing the $x[n]$ values

$$x_C(t) \rightarrow x_p(t) \rightarrow x[n] = x_C(nT)$$

C/D conversion

Conversion of impulse train to discrete-time sequence

$p(t)$$x_C(t)$$x_p(t)$$x[n] = x_C(nT)$

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Frequency-Domain Interpretation of C/D Conversion

\[ x_p(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t-nT) \]

\[ \Downarrow \mathcal{F} \]

\[ X_p(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j(\omega-k\omega_s)) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT} \quad (1) \]

CT – Periodic with period \( \frac{2\pi}{T} \)

\[ X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n} \quad (2) \]

DT – Periodic with period \( 2\pi \)
Frequency-Domain Interpretation of C/D Conversion

\[
\downarrow \quad \text{Compare Eqs. (1) & (2) and note} \quad \Omega = \omega T
\]

\[
X_d(e^{j\Omega}) = X_p\left(j\left(\frac{\Omega}{T}\right)\right)
\]

\[\text{Note: } \omega_s \Leftrightarrow 2\pi\]

\(CT \quad DT\)
Illustration of C/D Conversion in the Frequency-Domain

\[ X_c(j\omega) \]

\[ X_p(j\omega) \]

\[ X_d(e^{j\Omega}) \]

\[ T = T_1 \]

\[ T = T_2 = 2T_1 \]

\[ \Omega = \omega T_1 \]

\[ \Omega = \omega T_2 \]

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D/C Conversion $y_d[n] \rightarrow y_c(t)$
Reverse of the process of C/D conversion

$Y_p(j\omega) = Y_d(e^{j\omega T})$ — reverses frequency scaling

Again, $\Omega = \omega$

$$Y_c(j\omega) = \begin{cases} 
TY_d(e^{j\omega T}), & |\omega| < \frac{\omega_s}{2} \text{ — bandlimited} \\
0, & \text{otherwise}
\end{cases}$$
Now the whole picture

- Overall system is **time-varying** if sampling theorem is *not* satisfied
- It is LTI if the sampling theorem is *satisfied*, i.e. for bandlimited inputs $x_c(t)$, with
  $\omega_M < \frac{\omega_s}{2}$
- When the input $x_c(t)$ is band-limited ($X(j\omega) = 0$ at $|\omega| > \omega_M$) and the sampling theorem is satisfied ($\omega_s > 2\omega_M$), then

$$Y_c(j\omega) = H_c(j\omega)X_c(j\omega) \iff y_c(t) = h_c(t) * x_c(t) \quad LT1$$
Frequency-Domain Illustration of DT Processing of CT Signals

Sampling

CT freq->DT freq

Interpolate (LPF) -> Equivalent CT filter

DT filter

DT freq->CT freq

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Assuming No Aliasing

Step 1 - C/D: \( X_d \left( e^{j\Omega} \right) = X_p \left( j\Omega / T \right) \) - periodic

\[ N = \frac{\Omega}{T} = \frac{1}{T} X_c \left( j\Omega / T \right), \quad -\pi < \Omega < \pi \] if no aliasing

Step 2 - DT Filter: \( Y_d \left( e^{j\Omega} \right) = H_d \left( e^{j\Omega} \right) X_d \left( e^{j\Omega} \right) \)

\[ = \frac{1}{T} H_d \left( e^{j\Omega} \right) X_c \left( j\Omega / T \right), \quad -\pi < \Omega < \pi \]

Step 3 - D/C: \( Y_c \left( j\omega \right) = \begin{cases} T Y_d \left( e^{j\omega T} \right) = H_d \left( e^{j\omega T} \right) X_c \left( j\omega \right), & -\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2} \\ 0, & \text{otherwise} \end{cases} \)

\[ \Omega = \omega T \]

\[ H_c \left( j\omega \right) = \begin{cases} H_d \left( e^{j\omega T} \right), & |\omega| < \frac{\omega_s}{2} \\ 0, & \text{otherwise} \end{cases} \]

In practice, first specify the desired \( H_c(j\omega) \), then design \( H_d(e^{j\Omega}) \).
Example: Digital Differentiator

Applications: Edge Enhancement
Construction of Digital Differentiator

Bandlimited Differentiator

Desired: \( H_c(j\omega) = \begin{cases} j\omega, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases} \)

Set \( \omega_s = 2\omega_c \Rightarrow T = \frac{2\pi}{\omega_s} = \frac{\pi}{\omega_c} \)

Assume \( \omega_M < \omega_c \) (Nyquist rate met)

Choice for \( H_d(e^{j\Omega}) \):

\[
H_d(e^{j\Omega}) = \begin{cases} H_c(j\Omega/T), & |\Omega| < \pi \\ \text{periodic}, & |\Omega| \geq \pi \end{cases}
\]

\[
= j\left(\frac{\Omega}{T}\right) = j\omega_c\left(\frac{\Omega}{\pi}\right) \quad |\Omega| < \pi
\]
Band-Limited Digital Differentiator (continued)