Homework 2 (Stochastic Processes)

1. Briefly explain strong (strict sense) and weak (wide sense) stationarity, compare them and explain the relation of them.

2. Explain about properties, importance and applications of white noise.

3. Prove that for LTI systems, WSS input generates WSS output.

4. Answer the following questions about Gaussian processes:
   (a) Prove that If $X, Y$ have a bivariate normal distribution then $X$ and $Y$ are independent if and only if they are uncorrelated!
   (b) For Gaussian processes what can you say about relation of WSS and SSS? Why?

5. Which of the following functions can be covariance functions of a stationary process and which can not? Justify your answer!
   (a) $r(\tau) = \frac{\sin \tau}{\tau}$
   (b) $r(\tau) = \frac{\sin \tau}{\tau}$
   (c) $r(\tau) = \begin{cases} 1 & \tau = 0, \\ 2 & \tau = 1, \\ 0.5 & \tau = 2, \\ 0 & \text{o.w.} \end{cases}$
   (d) $r(\tau) = \cos \tau$
   (e) $r(\tau) = e^{\tau}$

6. Which conditions should the elements of the matrix
   \[ R = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]
   satisfy so that R could be a valid autocorrelation matrix of
   (a) A two-dimensional random vector?
   (b) A stationary scalar-valued stochastic process?

7. Assume that the inverse $R^{-1}_x$ of the correlation matrix of the $n$ dimensional column random vector $x$ exists, show that $E\{x^T R^{-1}_x x\} = n$.

8. Let $\{X(t), \in R\}$, be a stationary normal process with covariance function $r_X(t)$. Let $Y(t)$ be denoted by: $Y(t) = X(t) - 0.4X(t-2)$. Compute the covariance function of $\{Y(t)\}$. Is this process a Gaussian process? Is it stationary?

9. For Random walk process $X_t = X_{t-1} + \varepsilon_t, \varepsilon_t \sim WN(0, \sigma^2)$, what can you say about stationarity of $X_t$? What about $Y_t = X_t - X_{t-1}$?
10. For each of the following, determine whether the process is WSS and SSS.

(a) \( x(n) = A \cos(n \omega_0) \) where \( A \) is a Gaussian random variable with mean \( \mu_A \) and variance \( \sigma^2_A \).

(b) \( x(n) = A \cos(n \omega_0 + \varphi) \) where \( \varphi \) is a uniform random variable in \([-\pi, \pi]\), \( A \) is a constant.

(c) \( x(n) = A \cos(n \omega_0) + B \sin(n \omega_0) \) where \( A \) and \( B \) are uncorrelated zero mean random variables with variance \( \sigma^2 \).

(d) A Bernoulli process with \( P[x(n) = 1] = 1 - P[x(n) = -1] = p \).

(e) \( y(n) = x(n) - x(n - 1) \) where \( x(n) \) is a Bernoulli process.

11. Let \( C_i \) be binary iid random variables that get the values +1 and -1 with probability \( \frac{1}{2} \). In order to transmit binary symbols "1" and "0" through a communication channel we use rectangular pulses \( f(t) \) and \( -f(t) \).

\[
f(t) = \begin{cases} 
1 & 0 \leq t < 1 \\
0 & \text{o.w.}
\end{cases}
\]

The data transmitted through the channel is represented by the random process

\[
X(t) = \sum_{k=0}^{\infty} C_k f(t - k), \quad t \geq 0
\]

(a) Find first and second order pmfs of \( X(t) \).

(b) Find the mean and the autocorrelation of \( X(t) \).