Homework 3 (LTI systems)

1. (a) Discuss why we use power spectral density for frequency analysis of stochastic processes?
   (b) Why $R(\tau)$ give us some information about frequency domain?

2. if $y(t) = x(t + 2a) - x(t - 2a)$
   (a) find $R_{yy}(\tau)$.
   (b) Show that $S_{yy}(\omega) = 4S_{xx}(\omega)\sin^2(a\omega)$.

3. A WSS white noise process $W(t)$ is passed through a low-pass filter (LPF) with bandwidth $B/2$. Find the autocorrelation function of the output process.

4. Denote by $X(t)$ a real-valued continuous-time WSS random process with autocorrelation function $R_X(\tau)$. power spectral density of $X(t)$ is defined as $S_X(f)$, where $f$ is frequency
   (a) Show that the power spectral density $S_X(f)$ is real-valued.
   (b) Show that $S_X(f)$ is an even function.

5. Let $x(t)$ be a real valued, continuous time, zero mean WSS random process with correlation function $R_{xx}(\tau)$ and power spectrum $S_{xx}(\omega)$. Suppose $x(t)$ is the input to two real valued LTI systems as depicted below, producing two new processes $y_1(t)$ and $y_2(t)$. Find $C_{y_1,y_2}(\tau)$ and $S_{y_1,y_2}(\omega)$.

6. A time-continuous zero-mean Gaussian process $x(t)$ has power spectral density $R_x(f) = 2\pi e^{-\pi|f|}$, where $f$ is frequency. The process $y(t)$ is created by $y(t) = x(t) + 3x'(t) - 2x'(t - 1)$. Compute the probability that $y(t) > 2$.

7. The process $x(t)$ is WSS and $R_{xx}(\tau) = 5\delta(\tau)$.
   (a) Find $E\{y^2(t)\}$ and $S_{yy}(\omega)$ if $y'(t) + 3y(t) = x(t)$.
   (b) Find $E\{y^2(t)\}$ and $R_{xy}(t_1, t_2)$ if $y'(t) + 3y(t) = x(t)U(t)$ that $U(t)$ is step function. Sketch the functions $R_{xy}(2, t_2)$ and $R_{xy}(t_1, 3)$.
8. Consider an LTI system with system function:

\[ H(s) = \frac{1}{s^2 + 4s + 13} \]

The input to this system is a WSS process \( X(t) \) with \( E\{X^2(t)\} = 10 \).
Find \( S_X(\omega) \) such that the average power of output is maximum.

9. Consider an LTI system described by the following differential equation:

\[ \frac{d^2z}{dt^2} + 7\frac{dz}{dt} + 10z = x \]

There is a switch at the input of the system: there are two choices. In the first case an input \( z(t) \) with \( \mu_z = 0 \) and \( R_{zz}(\tau) = e^{-2|\tau|} \) will be connected to the input line. In the second case an input \( w(t) \) with \( \mu_w = 0 \) and \( R_{ww}(\tau) = 2e^{-|\tau|} \) will be connected to the input line. At the beginning (from \( t = -\infty \) to \( t = 0 \)) the switch is in the first case. At \( t = 0 \) the switch is set into the second case. Find the mean of \( y \) and \( R_{wy} \).

10. In below figure, you see three different random signals and their correlation and spectral densities. what figures are for original random signals, correlation functions and spectral densities? which of them are for the same process?

![Figure 1: random signals, correlation functions and Spectral densities.](image)

11. Problem 9-29 and 9-32 from Papoulis.