1) Using the bilinear transform frequency mapping equation,
\[ \Omega_c = \frac{2}{T} \tan \left( \frac{\omega_c + 2\pi k}{2} \right) = \frac{2}{T} \tan \left( \frac{\omega}{2} \right) \]
=> \[ T = \frac{2}{2\pi(300)} \tan \left( \frac{3\pi}{5} \right) = 1.46 \text{ms} \]

2) For impulse invariance method, we have:

\[ \Omega = \frac{\omega}{T_d} \quad , \quad H(e^{j\omega}) = H(j\Omega) \]

So,

(a) \[ 89125 \leq |H(j\Omega)| \leq 1, \quad 0 \leq |\Omega| \leq \frac{0.2\pi}{T_d} \]
\[ |H(j\Omega)| \leq 0.17783, \quad \frac{0.3\pi}{T_d} \leq |\Omega| \leq \frac{\pi}{T_d} \]

(b) Because the Butterworth frequency response is monotonic, we can solve:

\[ \left| H_c \left( \frac{j0.2\pi}{T_d} \right) \right|^2 = \frac{1}{1 + \left( \frac{0.2\pi}{\Omega_c T_d} \right)^{2N}} = 0.89125^2 \]
\[ \left| H_c \left( \frac{j0.3\pi}{T_d} \right) \right|^2 = \frac{1}{1 + \left( \frac{0.3\pi}{\Omega_c T_d} \right)^{2N}} = 0.17783^2 \]

To obtain \( \Omega_c T_d = 0.7047 \) and \( N = 5.88 \). Rounding up to \( N = 6 \) and yields \( \Omega_c T_d = 0.7032 \).

(c) We see that the poles of the magnitude-squared function are again evenly distributed around a circle radius 0.7032. For being causal and stable, \( H_c(s) \) should have the left half-plane poles of the magnitude squared function & the result is the same for any value of \( T_d \). Correspondingly, \( H(z) \) does not depend on \( T_d \).
3) (a) 

\[ 1 - \delta_1 = \frac{1 - \delta_1}{1 + \delta_1} \implies \delta_1 = \frac{2\delta_1}{1 + \delta_1} \]

\[ \delta_2 = \frac{\delta_2}{1 + \delta_1} \]

(b) 

\[ \delta_1 = \frac{\delta_1}{2 - \delta_1}, \quad \delta_2 = \frac{2\delta_2}{2 - \delta_2} \]

For the example, we were given 

\[ \hat{\delta}_1 = 1 - 0.89125 = 0.10875 \]

\[ \hat{\delta}_2 = 0.17783 \]

So:

\[ \delta_1 = 0.0575 \]

\[ \delta_2 = 0.1881 \]

(c) We can calculate \( H(z) \) as example 7.2.

4) Using the bilinear transform frequency mapping equation,

\[ \Omega_s = \frac{2}{T} \left( \frac{\omega_s}{2} \right) = \frac{2}{0.002} \tan \left( \frac{0.2\pi}{2} \right) = 2\pi(51.7126) \frac{rad}{s} \]

\[ \Omega_p = \frac{2}{T} \left( \frac{\omega_p}{2} \right) = \frac{2}{0.002} \tan \left( \frac{0.3\pi}{2} \right) = 2\pi(81.0935) \frac{rad}{s} \]

Thus, the specification which should be used to design the prototype continuous-time filter are

\[ |H_c(j\Omega)| < 0.04 \quad |\Omega| \leq 2\pi(51.7126) \]

\[ 0.995 < |H_c(j\Omega)| < 1.005 \quad |\Omega| \geq 2\pi(81.0935) \]
7.28. (a) We have

\[
\begin{align*}
s &= \frac{1 - z^{-1}}{1 + z^{-1}} \\
j\Omega &= \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \\
 &= \frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}} \\
\Omega &= \tan\left(\frac{\omega}{2}\right) \\
\Omega_p &= \tan\left(\frac{\omega_p/2}{2}\right) \quad \leftrightarrow \quad \omega_{p1} = 2\tan^{-1}(\Omega_p)
\end{align*}
\]

(b)

\[
\begin{align*}
s &= \frac{1 + z^{-1}}{1 - z^{-1}} \\
j\Omega &= \frac{1 + e^{-j\omega}}{1 - e^{-j\omega}} \\
 &= \frac{e^{j\omega/2} + e^{-j\omega/2}}{e^{j\omega/2} - e^{-j\omega/2}} \\
\Omega &= -\cot\left(\frac{\omega}{2}\right) \\
 &= \tan\left(\frac{\omega - \pi}{2}\right) \\
\Omega_p &= \tan\left(\frac{\omega_p - \pi}{2}\right) \quad \leftrightarrow \quad \omega_{p2} = \pi + 2\tan^{-1}(\Omega_p)
\end{align*}
\]

(c)

\[
\tan\left(\frac{\omega_{p2} - \pi}{2}\right) = \tan\left(\frac{\omega_{p1}}{2}\right)
\]

\[\Rightarrow \omega_{p2} = \omega_{p1} + \pi\]

(d)

\[H_2(z) = H_1(z)|_{z = -z}\]

The even powers of \(z\) do not get changed by this transformation, while the coefficients of the odd powers of \(z\) change sign.

Thus, replace \(A, C, 2\) with \(-A, -C, -2\).
7.29. (a) Substituting $Z = e^{j\theta}$ and $z = e^{j\omega}$ we get,

\[
e^{j\theta} = -e^{j2\omega} = e^{j(2\omega + \pi)} = e^{j2\theta}
\]

\[
\theta = 2\omega + \pi \quad \leftrightarrow \quad \omega = \frac{\theta - \pi}{2}
\]

(b) 

(c) 

\[
h[n] \quad \leftrightarrow \quad H(e^{j\theta})
\]

\[
h_1[n] \quad \leftrightarrow \quad H(e^{j(2\omega + \pi)})
\]

In the frequency domain, we first shift by $\pi$ and then we upsample by 2. In the time domain, we can write that as

\[
h_1[n] = \begin{cases} \frac{(-1)^n}{2} h[n/2], & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}
\]

(d) In general, a filter

\[
H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{M-1} z^{M-1} + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{N-1} + a_N z^{-N}}
\]

will transform under \(H_1(z) = H(-z^2)\) to

\[
H_1(z) = \frac{b_0 - b_1 z^{-2} + b_2 z^{-4} + \cdots - b_{M-1} z^{2M-2} + b_M z^{-2M}}{a_0 - a_1 z^{-2} + a_2 z^{-4} + \cdots - a_N z^{2N-2} + a_N z^{-2N}}
\]

where we are assuming here that $M$ and $N$ are even. All the delay terms increase by a factor of two, and the sign of the coefficient in front of any odd delay term is negated.

The given difference equations therefore become

\[
g[n] = x[n] + a_1 g[n - 2] - b_1 f[n - 4]
\]

\[
f[n] = -a_2 g[n - 2] - b_2 f[n - 2]
\]

\[
y[n] = c_1 f[n] + c_2 g[n - 2]
\]

To avoid any possible confusion please note that the $b_k$ and $a_k$ in these difference equations are not the same $b_k$ and $a_k$ shown above for the general case.
7.30. We are given

\[ H(z) = H_c(s) \bigg|_{s=\frac{1-z^{-\alpha}}{1+z^{-\alpha}}}, \]

where \(\alpha\) is a nonzero integer and \(\beta\) is a real number.

(a) It is true for \(\beta > 0\).

Proof:

\[
\begin{align*}
  s &= \beta \left[ \frac{1 - z^{-\alpha}}{1 + z^{-\alpha}} \right] \\
  s + sz^{-\alpha} &= \beta - \beta z^{-\alpha} \\
  s - \beta &= -\beta z^{-\alpha} - sz^{-\alpha} \\
  \beta - s &= z^{-\alpha}(\beta + s) \\
  z^{-\alpha} &= \frac{\beta - s}{\beta + s} \\
  z^\alpha &= \frac{\beta + s}{\beta - s}
\end{align*}
\]

The poles \(s_k\) of a stable, causal, continuous-time filter satisfy the condition \(\Re \{s\} < 0\). We want these poles to map to the points \(z_k\) in the \(z\)-plane such that \(|z_k| < 1\). With \(\alpha > 0\) it is also true that if \(|z_k| < 1\) then \(|z_k^\alpha| < 1\). Letting \(s_k = \sigma + j\omega\) we see that

\[
\begin{align*}
  |z_k| &< 1 \\
  |z_k^\alpha| &< 1 \\
  |\beta + \sigma + j\Omega| &< |\beta - \sigma - j\Omega| \\
  (\beta + \sigma)^2 + \Omega^2 &< (\beta - \sigma)^2 + \Omega^2 \\
  2\sigma\beta &< -2\sigma\beta
\end{align*}
\]

But since the continuous-time filter is stable we have \(\Re \{s_k\} < 0\) or \(\sigma < 0\). That leads to

\[-\beta < \beta\]

This can only be true if \(\beta > 0\).
(b) It is true for $\beta < 0$. The proof is similar to the last proof except now we have $|x_0| > 1$.

(c) We have

\[
    z^2 = \frac{1 + s}{1 - s} \bigg|_{x = j\Omega}
\]

\[
    |z^2| = 1
\]

\[
    |z| = 1
\]

Hence, the $j\Omega$ axis of the $s$-plane is mapped to the unit circle of $z$-plane.

(d) First, find the mapping between $\Omega$ and $\omega$.

\[
    j\Omega = \frac{1 - e^{-j2\omega}}{1 + e^{-j2\omega}} = \frac{e^{j\omega} - e^{-j\omega}}{e^{j\omega} + e^{-j\omega}}
\]

\[
    \Omega = \tan(\omega)
\]

\[
    \omega = \tan^{-1}(\Omega)
\]

Therefore,

\[
    1 - \delta_1 \leq |H(e^{j\omega})| \leq 1 + \delta_1, \quad \left\{ |\omega| \leq \frac{\pi}{4} \right\} \cup \left\{ \frac{3\pi}{4} < |\omega| < \pi \right\}
\]

Note that the highpass region $3\pi/4 \leq |\omega| \leq \pi$ is included because $\tan(\omega)$ is periodic with period $\pi$.

7.31. (a)

\[
    s = \frac{1 + z^{-1}}{1 - z^{-1}} \quad \leftrightarrow \quad z = \frac{s + 1}{s - 1}
\]

Now, we evaluate the above expressions along the $j\Omega$ axis of the $s$-plane

\[
    z = \frac{j\Omega + 1}{j\Omega - 1}
\]

\[
    |z| = 1
\]
(b) We want to show $|z| < 1$ if $\Re\{s\} < 0$.

\[ z = \frac{\sigma + j\Omega + 1}{\sigma + j\Omega - 1} \]

\[ |z| = \frac{\sqrt{(\sigma + 1)^2 + \Omega^2}}{\sqrt{(\sigma - 1)^2 + \Omega^2}} \]

Therefore, if $|z| < 1$

\[ (\sigma + 1)^2 + \Omega^2 < (\sigma - 1)^2 + \Omega^2 \]
\[ \sigma < -\sigma \]

It must also be true that $\sigma < 0$. We have just shown that the left-half $s$-plane maps to the interior of the $z$-plane unit circle. Thus, any pole of $H_c(s)$ inside the left-half $s$-plane will get mapped to a pole inside the $z$-plane unit circle.

(c) We have the relationship

\[ j\Omega = \frac{1 + e^{-j\omega}}{1 - e^{-j\omega}} = \frac{e^{j\omega/2} + e^{-j\omega/2}}{e^{j\omega/2} - e^{-j\omega/2}} \]
\[ \Omega = -\cot(\omega/2) \]

\[ |\Omega_s| = |\cot(\pi/6)| = \sqrt{3} \]
\[ |\Omega_P_1| = |\cot(\pi/2)| = 0 \]
\[ |\Omega_P_2| = |\cot(\pi/4)| = 1 \]

Therefore, the constraints are

\[ 0.95 \leq |H_c(j\Omega)| \leq 1.05, \quad 0 \leq |\Omega| \leq 1 \]
\[ |H_c(j\Omega)| \leq 0.01, \quad \sqrt{3} \leq |\Omega| \]