Stochastic Processes

Random Walks, Wiener process and Brownian motion

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Overview

- Reading Assignment
  - Chapter 10 of textbook

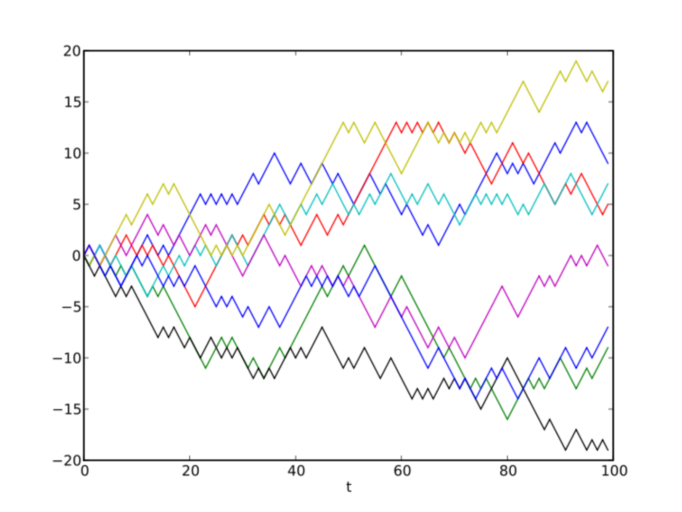
- Further Resources
  - MIT Open Course Ware
  - S. Karlin and H. M. Taylor, *A First Course in Stochastic Processes*, 2nd ed., Academic Press, New York, 1975.

Outline

- Random Walks
  - Useful properties of RWs
- Wiener Processes
- Brownian Motion

Basic Definitions

- In a RW model, a particle takes a unit step up or down at regular intervals, and $s_n$represents the position of particle at nth step.
- Examples:
  - The path traced by a molecule
  - The path of a drunk man
  - Gambler's wealth



Mathematical Definition

- Consider a sequence of i.i.d. random variables $x_1, x_2, \dots, x_n, \dots$ taking the values +1, -1 with probabilities p,q.
- s_n denotes the partial sum of n RVs :

$$s_n = x_1 + x_2 + \cdots + x_n \qquad s_0 = 0$$

- The RW is symmetric if $p = q = 0.5$

$$P\{s_n = r\} = \binom{n}{k} p^k q^{n-k}, \qquad k = \# + 1$$

$$\boldsymbol{P_{n,r} = \binom{n}{\frac{n+r}{2}} p^{\frac{n+r}{2}} q^{\frac{n-r}{2}}, \qquad r = 2k - n}$$

Probability of return

- We say the particle returned to origin at time t if $X(t) = 0$.
- For the particle to return to origin, it must do m up moves and m down moves.
- Lemma: the Probability of returning to origin at an odd time is 0

Probability of return (cont.)

- Probability of returning to origin at time 2m is:

$$u_{2m} = \binom{2m}{m} p^m q^m$$

Why?

First return

- Define event E as: The process touches zero for the first time at time 2m
- $v_{2m} \triangleq \Pr\{E\} = \frac{\#\ Paths\ from\ 0\ to\ 2m\ not\ touching\ zero\ except\ at\ endpoints}{Paths\ from\ 0\ to\ 2m}$

First return (Cont.)

- Theorem: For $n \geq 1$ the probabilities $\{u_{2k}\}$ and $\{v_{2k}\}$ are related by the equation

$$u_{2n} = v_0 u_{2n} + v_2 u_{2n-2} + \cdots + v_{2n} u_0$$

- Proof: Consider the n cases where the first return is at $1 \leq i \leq n$
- To solve by the generating functions method we define:

$$U(x) = \sum_{m=0}^{\infty} u_{2m} x^m, \mathrm{V}(x) = \sum_{m=0}^{\infty} v_{2m} x^m$$

First Return Solution

- From the recurrence relation:

$$U(x) = 1 + U(x)V(x)$$

From now on p = q = ½

- We also know that

$$\frac{1}{\sqrt{1-4x}} = \sum_{m=0}^{\infty} \binom{2m}{m} x^m$$

- Therefore $U(x) = \frac{1}{\sqrt{1-x}}$

First Return Solution (Cont.)

- $V(x) = \frac{U(x)-1}{U(x)} = 1 - (1-x)^{\frac{1}{2}} \Rightarrow V'(x) = \frac{U(x)}{2}$
- By integrating the series for U(x):

$$v_{2m} = \frac{u_{2m}}{(2m-1)}$$

Probability of Eventual Return

- Define w_n as the probability of the first return being no later than time n.
- We define the probability of eventual return as the limit of w_n in ∞:

$$w_* = \lim_{n\to\infty} w_n$$

- From definition we have:

$$w_{2n+1} = w_{2n} = \sum_{i=1}^{n} v_{2i} \Rightarrow w_* = \sum_{i=1}^{\infty} v_{2i}$$

Prob. of Eventual Return (Cont.)

- Substituting $x = 1$ in

$$\mathrm{V}(x) = \sum_{m=0}^{\infty} v_{2m} x^m$$

- We have $w_* = V(1)$
- Using the fact that

$$V(x) = \frac{U(x)-1}{U(x)} = 1 - (1-x)^{\frac{1}{2}}$$

We have $w_* = V(1) = 1$. Therefore the prob. of eventual return is 1.

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The Wiener Process

- T represents the duration of a step, and step size is $\pm$ s.
- $x(nT) = x_1 + x_2 + \cdots + x_n$
- Wiener Process: to study the limiting behavior of the random walk as $n \to \infty$, $T \to 0$ and $s \to 0$.

The Wiener Process

- $P\{X(nT)=ms\}=p\{\frac{n+m}{2}$ steps up and $\frac{n-m}{2}$ steps down$\}$

$$=\binom{n}{k}p^k q^{n-k} \qquad \text{where } k=\frac{n+m}{2}$$

- if n is large and k is in the $\sqrt{vpn}$ vicinity on np then:

$$\binom{n}{k}p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}}e^{-(k-np)^2/2\pi pq}$$

$$p=q=0.5 \Rightarrow P\{X(nT)=ms\} \approx \frac{1}{\sqrt{np/2}}e^{-m^2/2\pi} \qquad m \in O\left(\sqrt{n}\right)$$

The Wiener Process

- $P\{X(nT)=ms\} \approx \frac{1}{\sqrt{np/2}} e^{-m^2/2\pi}$

  $E[X(nT)]=0 \qquad E[X^2(nT)]=ns^2$

- as $n \to \infty$ and $T \to 0$ if we set $t=nT$ then:

  $P\{X(t) \le ms\} \approx G(m/\sqrt{n})$ G(x) is the N(0,1) distribution

  $E[X(t)]=0 \quad E[X^2(t)]=ns^2=\frac{ts^2}{T}$

  $W(t)=\lim_{T\to\infty}(X(t)) \qquad E[W(t)]=0$

  assume that s tends to 0 as $\sqrt{T}$: $s^2=\alpha T \Rightarrow E[W^2(t)=\alpha t]$

The Wiener Process (pdf)

$$P\{X(t) \le ms\} \approx G(m/\sqrt{n}) \qquad w = ms = m\sqrt{\alpha T}$$

$$n = t/T$$

$$P\{W(t) \le w\} = G\left(\frac{w/\sqrt{\alpha T}}{\sqrt{t/T}}\right) = G\left(\frac{w}{\sqrt{\alpha t}}\right)$$

$$\Rightarrow f_W(t) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-w^2/2\alpha t}$$

The Wiener Process (autocorrelation)

- Note: if $n_1 < n_2 \le n_3 < n_4$ then $X(n_4T) - X(n_3T)$ and $X(n_2T) - X(n_1T)$ are independent.

  so if $t_1 < t_2$ then $W(t_2) - W(t_1)$ and $W(t_1)$ are independent.

$$E\{[W(t_2) - W(t_1)]W(t_1)\} = E\{[W(t_2) - W(t_1)]\}E\{W(t_1)\} = 0$$

$$\Rightarrow E\{W(t_1)W(t_2)\} = E\{W^2(t_1)\} = \alpha t_1$$

$$R(t_1, t_2) = \alpha \min(t_1, t_2)$$

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Brownian Motion

To describe movement of a particle in a liquid:

$$mx''(t) + fx'(t) + cx(t) = F(t)$$

m: mass of the particle

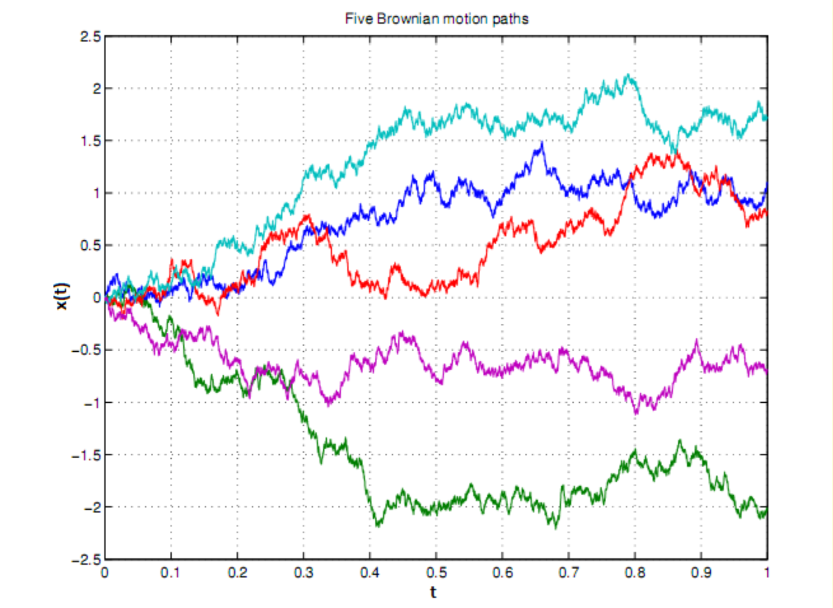
f: friction coefficient

cx(t): external force

F(t): collision force

x(t): particle's position

F(t), x(t): Random



Brownian Motion(Cont'd)

$$mx''(t) + fx'(t) + cx(t) = F(t)$$

F(t): Normal white Noise with zero mean and:

$$R_F(\tau) = 2kTf\delta(\tau), \qquad S_F(\omega) = 2kTf \;\; (constant)$$

$$S_X(\omega) = \frac{2kTf}{(c - m\omega^2)^2 + f^2\omega^2}$$

Solutions:

$$s_{1,2} = -\alpha \pm j\beta$$

$$R_X(\tau) = \frac{kT}{c} e^{-\alpha|\tau|}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta|\tau|)$$

So $$f_X(x) = \sqrt{\frac{c}{2\pi kT}}e^{-\frac{cx^2}{2kT}}$$

Is it WSS??

Free Motion

BM without External force

$$mx''(t) + fx'(t) = F(t)$$

$$mv'(t) + fv(t) = F(t), \qquad v(t) = x'(t)$$

$$S_v(\omega) = \frac{2kTf}{m^2\omega^2 + f^2}$$

$$R_v(\tau) = \frac{kT}{m} e^{-\frac{f|\tau|}{m}}$$

So $$f_v(v) = \sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{cv^2}{2kT}}$$

What about x(t)??

Free Motion(Cont'd)

$$x(t) = \int_0^t v(t)dt$$

So $$E[X^2(t)] = \frac{2kT}{m}\left(t - \frac{m}{f} + \frac{m}{f}e^{-\frac{ft}{m}}\right)$$

$$E[X^2(t)] = \frac{2kT}{m}t, \qquad t \gg \frac{m}{f}$$

In this case BM can be considered as a WP