

Stochastic Processes

Random Walks, Wiener process and Brownian motion



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Overview

□ Reading Assignment

- Chapter 10 of textbook

□ Further Resources

- MIT Open Course Ware
- S. Karlin and H. M. Taylor, *A First Course in Stochastic Processes*, 2nd ed., Academic Press, New York, 1975.



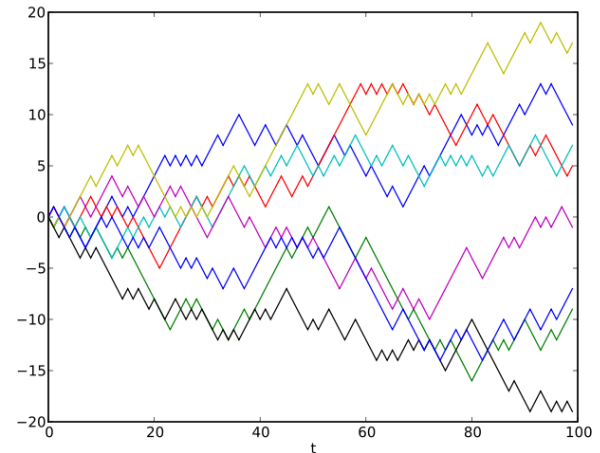
Outline

- Random Walks
 - Useful properties of RWs
- Wiener Processes
- Brownian Motion



Basic Definitions

- In a RW model, a particle takes a unit step up or down at regular intervals, and s_n represents the position of particle at n th step.
- Examples:
 - The path traced by a molecule
 - The path of a drunk man
 - Gambler's wealth



Mathematical Definition

- Consider a sequence of i.i.d. random variables $x_1, x_2, \dots, x_n, \dots$ taking the values $+1, -1$ with probabilities p, q .

- s_n denotes the partial sum of n RVs :

$$s_n = x_1 + x_2 + \dots + x_n \quad s_0 = 0$$

- The RW is symmetric if $p = q = 0.5$

$$P\{s_n = r\} = \binom{n}{k} p^k q^{n-k}, \quad k = \# + 1$$

$$P_{n,r} = \binom{n}{\frac{n+r}{2}} p^{\frac{n+r}{2}} q^{\frac{n-r}{2}}, \quad r = 2k - n$$



Probability of return

- We say the particle returned to origin at time t if $X(t) = 0$.
- For the particle to return to origin, it must do m up moves and m down moves.
- Lemma: the Probability of returning to origin at an odd time is 0



Probability of return (cont.)

- Probability of returning to origin at time $2m$ is:

$$u_{2m} = \binom{2m}{m} p^m q^m$$

Why?



First return

- Define event E as: The process touches zero for the first time at time $2m$
- $v_{2m} \triangleq \Pr\{E\} = \frac{\# \text{ Paths from } 0 \text{ to } 2m \text{ not touching zero except at endpoints}}{\text{Paths from } 0 \text{ to } 2m}$



First return (Cont.)

- Theorem: For $n \geq 1$ the probabilities $\{u_{2k}\}$ and $\{v_{2k}\}$ are related by the equation

$$u_{2n} = v_0 u_{2n} + v_2 u_{2n-2} + \cdots + v_{2n} u_0$$

- Proof: Consider the n cases where the first return is at $1 \leq i \leq n$

- To solve by the generating functions method we define:

$$U(x) = \sum_{m=0}^{\infty} u_{2m} x^m, \quad V(x) = \sum_{m=0}^{\infty} v_{2m} x^m$$



First Return Solution

- From the recurrence relation:

$$U(x) = 1 + U(x)V(x)$$

From now on $p = q = \frac{1}{2}$

- We also know that

$$\frac{1}{\sqrt{1-4x}} = \sum_{m=0}^{\infty} \binom{2m}{m} x^m$$

- Therefore $U(x) = \frac{1}{\sqrt{1-x}}$



First Return Solution (Cont.)

$$\square V(x) = \frac{U(x)-1}{U(x)} = 1 - (1-x)^{\frac{1}{2}} \Rightarrow V'(x) = \frac{U(x)}{2}$$

\square By integrating the series for $U(x)$:

$$v_{2m} = \frac{u_{2m}}{(2m-1)}$$



Probability of Eventual Return

- Define w_n as the probability of the first return being no later than time n .
- We define the probability of eventual return as the limit of w_n in ∞ :

$$w_* = \lim_{n \rightarrow \infty} w_n$$

- From definition we have:

$$w_{2n+1} = w_{2n} = \sum_{i=1}^n v_{2i} \Rightarrow w_* = \sum_{i=1}^{\infty} v_{2i}$$



Prob. of Eventual Return (Cont.)

- Substituting $x = 1$ in

$$V(x) = \sum_{m=0}^{\infty} v_{2m} x^m$$

- We have $w_* = V(1)$

- Using the fact that

$$V(x) = \frac{U(x)-1}{U(x)} = 1 - (1-x)^{\frac{1}{2}}$$

We have $w_* = V(1) = 1$. Therefore the prob. of eventual return is 1.



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The Wiener Process

- T represents the duration of a step, and step size is $\pm s$.
- $x(nT) = x_1 + x_2 + \cdots + x_n$
- Wiener Process: to study the limiting behavior of the random walk as $n \rightarrow \infty$, $T \rightarrow 0$ and $s \rightarrow 0$.



The Wiener Process

- $P\{X(nT) = ms\} = p\left\{\frac{n+m}{2} \text{ steps up and } \frac{n-m}{2} \text{ steps down}\right\}$

$$= \binom{n}{k} p^k q^{n-k} \quad \text{where } k = \frac{n+m}{2}$$

- if n is large and k is in the \sqrt{npn} vicinity on np then:

$$\binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)^2 / 2npq}$$

$$p = q = 0.5 \Rightarrow P\{X(nT) = ms\} \approx \frac{1}{\sqrt{np/2}} e^{-m^2 / 2\pi} \quad m \in O(\sqrt{n})$$



The Wiener Process

- $P\{X(nT) = ms\} \approx \frac{1}{\sqrt{np/2}} e^{-m^2/2\pi}$

$$E[X(nT)] = 0 \quad E[X^2(nT)] = ns^2$$

- as $n \rightarrow \infty$ and $T \rightarrow 0$ if we set $t = nT$ then:

$$P\{X(t) \leq ms\} \approx G(m/\sqrt{n}) \quad G(x) \text{ is the } N(0,1) \text{ distribution}$$

$$E[X(t)] = 0 \quad E[X^2(t)] = ns^2 = \frac{ts^2}{T}$$

$$W(t) = \lim_{T \rightarrow \infty} X(t) \quad E[W(t)] = 0$$

assume that s tends to 0 as $\sqrt{T} : s^2 = \alpha T \Rightarrow E[W^2(t)] = \alpha t$



The Wiener Process (pdf)

$$P\{X(t) \leq ms\} \approx G(m / \sqrt{n}) \quad \begin{array}{l} w = ms = m\sqrt{\alpha T} \\ n = t/T \end{array}$$

$$P\{W(t) \leq w\} = G\left(\frac{w / \sqrt{\alpha T}}{\sqrt{t/T}}\right) = G\left(\frac{w}{\sqrt{\alpha t}}\right)$$



$$\Rightarrow f_W(t) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-w^2 / 2\alpha t}$$

The Wiener Process (autocorrelation)

- Note: if $n_1 < n_2 \leq n_3 < n_4$ then $X(n_4T) - X(n_3T)$ and $X(n_2T) - X(n_1T)$ are independent.
so if $t_1 < t_2$ then $W(t_2) - W(t_1)$ and $W(t_1)$ are independent.

$$E\{[W(t_2) - W(t_1)]W(t_1)\} = E\{[W(t_2) - W(t_1)]\}E\{W(t_1)\} = 0$$

$$\Rightarrow E\{W(t_1)W(t_2)\} = E\{W^2(t_1)\} = \alpha t_1$$

$$R(t_1, t_2) = \alpha \min(t_1, t_2)$$



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Brownian Motion

To describe movement of a particle in a liquid:

$$mx''(t) + fx'(t) + cx(t) = F(t)$$

m: mass of the particle

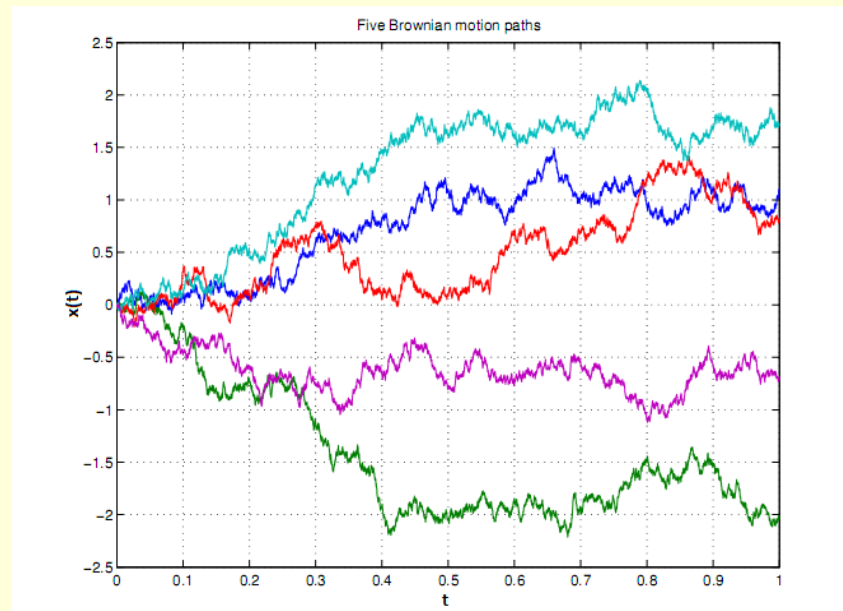
f: friction coefficient

cx(t): external force

F(t): collision force

x(t): particle's position

F(t), x(t): Random



Brownian Motion(Cont'd)

$$mx''(t) + fx'(t) + cx(t) = F(t)$$

$F(t)$: Normal white Noise with zero mean and:

$$R_F(\tau) = 2kTf\delta(\tau), \quad S_F(\omega) = 2kTf \text{ (constant)}$$

$$S_X(\omega) = \frac{2kTf}{(c - m\omega^2)^2 + f^2\omega^2}$$

$$R_X(\tau) = \frac{kT}{c} e^{-\alpha|\tau|} \left(\cos \beta\tau + \frac{\alpha}{\beta} \sin \beta|\tau| \right)$$

Solutions:

$$s_{1,2} = -\alpha \pm j\beta$$

So $f_X(x) = \sqrt{\frac{c}{2\pi kT}} e^{-\frac{cx^2}{2kT}}$

Is it WSS??



Free Motion

BM without External force

$$mx''(t) + fx'(t) = F(t)$$

$$mv'(t) + fv(t) = F(t), \quad v(t) = x'(t)$$

$$S_v(\omega) = \frac{2kTf}{m^2\omega^2 + f^2}$$

$$R_v(\tau) = \frac{kT}{m} e^{-\frac{f|\tau|}{m}}$$

So $f_v(v) = \sqrt{\frac{m}{2\pi kT}} e^{-\frac{cv^2}{2kT}}$

What about $x(t)$??



Free Motion(Cont'd)

$$x(t) = \int_0^t v(t)dt$$

So
$$E[X^2(t)] = \frac{2kT}{m} \left(t - \frac{m}{f} + \frac{m}{f} e^{-\frac{ft}{m}} \right)$$

$$E[X^2(t)] = \frac{2kT}{m} t, \quad t \gg \frac{m}{f}$$

In this case BM can be considered as a WP

