1. Let $X$ and $Y$ be random variables with the joint probability density function:

\[
f(x, y) = \begin{cases} 
(k^2)(1 - x) & \text{if } 0 \leq y, x \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

- Find a value for $k$ such that $f(x, y)$ is a proper probability density function. Is there more than one value for $k$?
- Find the marginal distributions for $X$ and $Y$.
- Are $X$ and $Y$ independent?
- Find the Variance of these two random variables.

2. From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75.

- Give an upper bound for the probability that a student’s test score will exceed 85. Suppose, in addition, that the professor knows that the variance of a student’s test score is equal to 25.
- What can be said about the probability that a student will score between 65 and 85?
- How many students would have to take the examination to ensure, with probability at least .9, that the class average would be within 5 of 75? Do not use the central limit theorem.
- Use the central limit theorem to solve previous part.

3. Suppose a fair coin is tossed 1000 times. If the first 100 tosses all result in heads, what proportion of heads would you expect on the final 900 tosses?

4. Let $x_1, x_2, \ldots, x_n$ be a random samples from a Bernoulli distribution with parameter $p$:
   a) Formulate an estimator for $p$.
   b) Calculate the expected value of this estimator then, show that this estimator is unbiased.
   c) Calculate the variance of this estimator then, show that, when $n \to \infty$, this estimator is consistent.

5. Explain briefly:
   - The difference between maximum-likelihood estimation and maximum a posteriori estimation.
   - Define Bayes estimator $\hat{\theta}$ then, consider $J(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$ (Minimum mean square) as a risk function. Formulate Bayes estimator using this risk function.

6. Suppose, we have two independent sonar measurements $(z_1, z_2)$ of position $x$. The sensor error may be modeled as $P(z_1|x) = N(x, 10^2)$ and $P(z_2|x) = N(x, 20^2)$. By measuring, we obtained sensor readings of $z_1 = 130$ and $z_2 = 170$. Considering these reading, determine the maximum-likelihood estimator of $x$. 

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7. a) For the previous question, suppose in addition we have prior information that

\[ P(x) \sim N(150, 30^2) \]

determine the maximum a posteriori estimator for \( x \).

b) Suppose

\[ \sigma_L^2 = \sigma_1^2 + \sigma_2^2 \]

Compare MAP and ML estimators, then consider the following cases and explain them:
I) If \( \sigma_L \) is large
II) If \( \sigma_p \) is large

8. Let \( x_1, x_2, \ldots, x_n \) be i.i.d. random variables with the following PDF:

\[
  f_x(x; \theta) = \begin{cases} 
  e^{-(x-\theta)} & x \geq \theta \\
  0 & x < \theta 
  \end{cases}
\]

Where \( \theta \) is an unknown parameter. Find the maximum-likelihood estimate of \( \theta \).

9. For the previous question, suppose prior probability of \( \theta \) has the following distribution:

\[
  f_\theta(\theta) = \begin{cases} 
  e^{\theta-\alpha} & \theta \leq \alpha \\
  0 & \theta > \alpha 
  \end{cases}
\]

Find the MAP estimate for \( \theta \).

10. Let \( x_1, x_2, \ldots, x_n \) be a random samples from the exponential distribution with parameter \( \theta \). Assume a \( \text{Gama}(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \) prior for \( \theta \) and formulate a Bayes estimator for \( \theta \).

11. Let \( x_1, x_2, \ldots, x_n \) be a random samples from the Gaussian distribution \( N(\mu, \sigma^2) \).

Maximum-likelihood estimation gives us:

\[
  \hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}
\]

\[
  \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (x_i - \hat{\mu})^2}{n}
\]

Are these estimators unbiased? if not, how can we unbiased them?