Homework 2 (Stochastic Processes - Definitions)  
(225 + 30 points)

1. Reading assignment:
   - Section 9-1 from Papoulis.

2. (15 pts) Answer briefly to each of the following questions:
   (a) Give an example of a natural phenomenon that can be modeled as a SSS process.
   (b) Give an example of a natural phenomenon that can be modeled as a WSS process.
   (c) Give an example of a natural phenomenon that can be modeled as a non-WSS process.
   (d) Is it possible to say that a process is WSS from a sample path of that process?
   (e) Is every SSS process an IID stochastic process?

3. (15 pts) Which of the followings can be autocorrelation function of a WSS process. Prove your answer.
   (a) \( R_x(\tau) = A \sin(\omega \tau) \)
   (b) \( R_x(\tau) = A \cos(\omega \tau) \)
   (c) \( R_x(\tau) = A e^{\tau} \)
   (d) \( R_x(\tau) = A \delta(\tau) \)

4. (20 pts) Consider following SPs:
   \[ X_1(t) = A \sin(\omega t) + B \cos(\omega t) \]
   \[ X_2(t) = A \sin(\omega t) - B \cos(\omega t) \]
   \[ X_3(t) = B \sin(\omega t) - A \cos(\omega t) \]
   where A and B are random variables which are uncorrelated and have 0 mean and equal variance.
   (a) (10 pts) Are they WSS?
   (b) (5 pts) Are \( X_1 \) and \( X_2 \) jointly WSS?
   (c) (5 pts) Are \( X_1 \) and \( X_3 \) jointly WSS?
5. (35 pts) **Poisson Process**

A stochastic process \( \{N(t), t \geq 0\} \) is said to be a *counting process* if \( N(t) \) represents the total number of *events* that occur by time \( t \). For example if \( N(t) \) equals the number of goals that a given soccer player scores by time \( t \), then \( \{N(t), t \geq 0\} \) is a counting process. An event of this process will occur whenever the soccer player scores a goal. From its definition we see that for a counting process \( N(t) \) must satisfy:

i. \( N(t) \geq 0 \).

ii. \( N(t) \) is integer valued.

iii. If \( s < t \), then \( N(s) \leq N(t) \).

iv. For \( s < t \), \( N(t) - N(s) \) equals the number of events that occur in the interval \((s, t]\).

A counting process is said to possess *independent increments* if the numbers of events that occur in disjoint time intervals are independent. The counting process \( \{N(t), t \geq 0\} \) is said to be a *Poisson process* having rate \( \lambda, \lambda > 0 \), if

i. \( N(0) = 0 \).

ii. The process has independent increments.

iii. If \( s < t \), then \( N(s) \leq N(t) \).

iv. The number of events in any interval of length \( t \) is Poisson distributed with mean \( \lambda t \). That is for all \( s, t \geq 0 \):

\[
P[N(t + s) - N(s) = n] = e^{-\lambda t} \frac{e^{\lambda s}}{n!}, n = 0, 1, \ldots
\]

A counting process is said to possess *stationary increments* if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval. Note that it follows from condition (iii) that a Poisson process has stationary increments.

(a) (6 pts) Find autocorrelation function of this process.

(b) (6 pts) Show that if \( N_1(t) \) and \( N_2(t) \) are two Poisson processes with parameters \( \lambda_1 \) and \( \lambda_2 \), then \( N(t) = N_1(t) + N_2(t) \) is a Poisson process and find its parameter.

(c) (6 pts) Show that \( N(t) = N_1(t) - N_2(t) \) is not a Poisson process.

(d) (6 pts) If we tag each occurrence with probability \( p \) and then define a counting process which counts tagged points, show that this process is also a Poisson process.

(e) (6 pts) Show that the probability distribution of the waiting time until the next occurrence is an exponential distribution.

(f) (5 pts) Define renewal process and create one from previous part.
6. **Gaussian Process**

An important class of random processes is that of *Gaussian* or *normal processes*. Define this process and show that in the case of a normal process, being WSS implies being SSS.

7. (28 pts) Consider SP defined as:

\[
\{ X[n, s] : n \in \mathbb{W}, s \in \Omega \}
\]

where \( \mathbb{W} = \{0, 1, 2, \ldots\} \) and for each \( n \), \( X[n] \) is a random variable over sample space \( \Omega = \{1, 2, 3\} \). Also we now that:

\[
\forall s \in \Omega : P\{\{s\}\} = 1/3
\]

If we have:

\[
X[n, 1] = \begin{cases} 
3 & n = 0 \\
0 & n \neq 0 
\end{cases}
\]

\[
X[n, 2] = \begin{cases} 
1 & n > 0 \\
0 & n \leq 0 
\end{cases}
\]

\[
X[n, 3] = \cos\left(\frac{n\pi}{2}\right)
\]

(a) (12 pts) Find the mean of this process.

(b) (12 pts) Find \( R_x[m, n] \).

(c) (4 pts) Find \( m \) and \( n \) for which \( X[m] \) and \( X[n] \) are independent.

8. (25+10 pts) Let \( X_n, n \geq 0 \) be a Normal white noise process, that is, \( X_t \sim N(0,1) \) and \( X_t \)s are independent. Consider process \( Y_n, n \geq 1 \) with \( Y_0 = 0 \), and \( Y_n = Y_{n-1} + X_n \) for \( n \geq 1 \)

(a) (3 pts) Is \( Y_n \) have property of independent increments?

(b) (2 pts) Find \( \mathbb{E}\{Y_n\} \).

(c) (3 pts) Find \( R_Y(m, n) \).

(d) (5 pts) Define Markov process and prove \( Y_n \) is a Markov process.

(e) (10 extra pts) Prove \( Y_n \) is a Gaussian process.

(f) (4 pts) Find first order and second order pdfs of \( Y_n \).

(g) (4 pts) Find joint distribution of \( Y_1, Y_2 \) and \( Y_n \).

(h) (4 pts) Find \( \mathbb{E}\{Y_5|Y_1 = y_1, Y_2 = y_2\} \).

9. (5 pts) Suppose \( X(t) \) is a white Gaussian noise. Find autocorrelation function of

\[
Y(t) = \int_0^t X(s)ds
\]
10. (50 pts) Solve problems 12, 13, 20, 22, 27 from chapter 9 of Papoulis's book. (Each has 10 pts)

11. (20 + 20 pts) **Programming Assignment**
   Consider the following random processes:

   function V = sp1(m,n)
   M1 = ones(m,1)*6*sin([1:n]*pi/50);
   M2 = .03*ones(m,1)*[1:n];
   V = (rand(m,n)-.5).*M1+M2;

   function V = rp2(m,n)
   M1 = rand(m,1)*ones(1,n);
   M2 = rand(m,1)*ones(1,n);
   V = ((rand(m,n)-.5)*.5).*M2+M1;

   function V = rp3(m,n)
   V = (rand(m,n)-.5)*4+.4;

   Where argument \( |m| \) denotes number of simulations (sample paths) and \( |n| \) indicates time interval \([1, n]\) in which process is simulated.

   (a) (10 pts) For each process plot the sample path for 4 simulations and argue which of processes can be WSS.

   (b) (10 pts) Verify your answer to previous section analytically.

   (c) (20 extra pts) Estimate the process mean value for time points: 1, 10, 20, 30, 40, 50, 60. To accomplish better estimations you can increase number of simulations. Plot your estimated values against number of simulations. Also try to compare your results with analytical means.