1. The conditional covariance of $X$ and $Y$, given $Z$, is defined by:

$$\text{Cov}(X, Y|Z) = E[(X - E[X|Z])(Y - E[Y|Z])|Z]$$

- Show that:

$$\text{Cov}(X, Y|Z) = E[XY|Z] - E[X|Z]E[Y|Z]$$

- Prove the conditional covariance formula:

$$\text{Cov}(X, Y) = E[\text{Cov}(X, Y|Z)] + \text{Cov}(E[X|Z], E[Y|Z])$$

- Prove the conditional variance formula:

$$\text{Var}(X) = \text{Var}(E[X|Y]) + E[\text{Var}(X|Y)]$$

2. For jointly defined random variables $X$, $Y$ and $Z$ prove the followings:

- $E[XY] = E[XE[Y|X]]$
- With the assumption that $E[Y|X] = 1$ prove that:

$$\text{Var}(XY) \geq \text{Var}(X)$$

3. Calculate pdf of $X + Y$ when $X$ and $Y$ are independent and come from the following distributions:

- $X$ and $Y$ are uniformly distributed over $(0, 1)$.
- $X$ and $Y$ are from gamma distribution with parameters $(\alpha_1, \beta)$ and $(\alpha_2, \beta)$ respectively.

4. Let $X$ and $Y$ be independent random variables with probability density function

$$f_X(x) = \begin{cases} \beta^{-\alpha} x^{\alpha - 1} & 0 < x < \beta \\ 0 & \text{o.w.} \end{cases} \quad (\alpha > 1, \beta > 0)$$

- Let $Z = \min\{X, Y\}$ and $W = \max\{X, Y\}$. Find the joint density function of $Z, W$.
- Prove that $Z$ and $W$ are independent random variables.
5. Let $X, Y \sim N(0, \sigma^2)$. Let also $U$ and $V$ be two random variables such that:

\[ U = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}}, \quad V = \frac{XY}{\sqrt{X^2 + Y^2}} \]

- Find the joint density function of $U$ and $V$.
- Prove that $U$ and $V$ are normally distributed independent random variables.

6. Let $X$ be a random variable such that $E[X] = E[X^2] = 0$. Show that $P(X = 0) = 1$.

7. You have invited 64 guests to a party. You need to make sandwiches for the guests. You believe that a guest might need 0, 1 or 2 sandwiches with probabilities $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$ respectively. You assume that the number of sandwiches each guest needs is independent from other guests. How many sandwiches should you make so that you are 95% sure that there is no shortage?

8. Let $X_1, X_2, \ldots, X_n \sim U(0, 1)$. Assuming that

\[ M_n = \frac{\sum_{i=1}^{n} X_i}{n} \]

- Compute $E[M_n]$ and $Var(M_n)$.
- Find an upper bound for $P(|M_n - \frac{1}{2}| \geq \frac{1}{100})$.
- Find

\[ \lim_{n \to \infty} P(|M_n - \frac{1}{2}| \geq \frac{1}{100}) \]