Problem 1.

Compute the Fourier transform of each of the following signals:

(a) \( x[n] = u[n - 2] - u[n - 6] \)
(b) \( x[n] = \left(\frac{1}{2}\right)^{-n}u[-n - 1] \)
(c) \( x[n] = (\frac{1}{3})^{n}|u[-n - 2] \)
(d) \( x[n] = 2^n \sin(\frac{\pi}{4} n)u[-n] \)
(e) \( x[n] = (\frac{1}{2})^{n}\cos(\frac{\pi}{8}(n - 1)) \)
(f) \( x[n] = \begin{cases} n, & -3 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases} \)
(g) \( x[n] = \sin(\frac{\pi}{2}n) + \cos(n) \)
(h) \( x[n] = \sin(\frac{5\pi}{3}n) + \cos(\frac{7\pi}{3}n) \)
(i) \( x[n] = x[n - 6] \), and \( x[n] = u[n] - u[n - 5] \) for \( 0 \leq n \leq 5 \)
(j) \( x[n] = (n - 1)(\frac{1}{3})^{n} \)

Problem 2.
Let $X(e^{j\omega})$ denote the Fourier transform of the signal $x[n]$ depicted in Figure P5.23. Perform the following calculations without explicitly evaluating $X(e^{j\omega})$:

(a) Evaluate $X(e^{j0})$.

(b) Find $\Re X(e^{j\omega})$.

(c) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.

(d) Find $X(e^{j2\pi})$.

(e) Determine and sketch the signal whose Fourier transform is $\Re \{x(\omega)\}$.

(f) Evaluate:

(i) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$

(ii) $\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega$
Determine which, if any, of the following signals have Fourier transforms that satisfy each of the following conditions:

1. \( \Re\{X(e^{j\omega})\} = 0 \).
2. \( \Im\{X(e^{j\omega})\} = 0 \).
3. There exists an integer \( \alpha \) such that \( e^{j\alpha\omega} X(e^{j\omega}) \) is real.
4. \( \int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 0 \).
5. \( X(e^{j\omega}) \) periodic.
6. \( X(e^{j\omega}) = 0 \).

(a) \( x[n] \) as in Figure P5.24(a)
(b) \( x[n] \) as in Figure P5.24(b)
(c) \( x[n] = (\frac{1}{2})^n u[n] \)
(d) \( x[n] = (\frac{1}{2})^{|n|} \)
(e) \( x[n] = \delta[n - 1] + \delta[n + 2] \)
(f) \( x[n] = \delta[n - 1] + \delta[n + 3] \)
(g) \( x[n] \) as in Figure P5.24(c)
(h) \( x[n] \) as in Figure P5.24(d)
(i) \( x[n] = \delta[n - 1] - \delta[n + 1] \)
Fig P5.24
Problem 4.

Let $x_1[n]$ be the discrete-time signal whose Fourier transform $X_1(e^{j\omega})$ is depicted in Figure P5.26(a).

(a) Consider the signal $x_2[n]$ with Fourier transform $X_2(e^{j\omega})$, as illustrated in Figure P5.26(b). Express $x_2[n]$ in terms of $x_1[n]$. \textit{[Hint: First express $X_2(e^{j\omega})$ in terms of $X_1(e^{j\omega})$, and then use properties of the Fourier transform.]}

(b) Repeat part (a) for $x_3[n]$ with Fourier transform $X_3(e^{j\omega})$, as shown in Figure P5.26(c).

(c) Let

$$\alpha = \frac{\sum_{n=-\infty}^{\infty} nx_1[n]}{\sum_{n=-\infty}^{\infty} x_1[n]}.$$

This quantity, which is the center of gravity of the signal $x_1[n]$, is usually referred to as the \textit{delay time} of $x_1[n]$. Find $\alpha$. (You can do this without first determining $x_1[n]$ explicitly.)
Fig P5.26a

Fig P5.26b,c
Problem 5.

The signals \( x[n] \) and \( g[n] \) are known to have Fourier transforms \( X(e^{j\omega}) \) and \( G(e^{j\omega}) \), respectively. Furthermore, \( X(e^{j\omega}) \) and \( G(e^{j\omega}) \) are related as follows:

\[
\frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\theta})G(e^{j(\omega-\theta)})d\theta = 1 + e^{-j\omega} \quad (P5.28-1)
\]

(a) If \( x[n] = (-1)^n \), determine a sequence \( g[n] \) such that its Fourier transform \( G(e^{j\omega}) \) satisfies eq. (P5.28–1). Are there other possible solutions for \( g[n] \)?

(b) Repeat the previous part for \( x[n] = (\frac{1}{2})^n u[n] \).

Problem 6.

Consider a causal LTI system described by the difference equation

\[
y[n] + \frac{1}{2} y[n - 1] = x[n].
\]

(a) Determine the frequency response \( H(e^{j\omega}) \) of this system.

(b) What is the response of the system to the following inputs?

(i) \( x[n] = (\frac{1}{2})^n u[n] \)

(ii) \( x[n] = (\frac{1}{2})^n u[n] \)

(iii) \( x[n] = \delta[n] + \frac{1}{2}\delta[n - 1] \)

(iv) \( x[n] = \delta[n] - \frac{1}{2}\delta[n - 1] \)

Problem 7.
Let $x[n]$ and $h[n]$ be two signals, and let $y[n] = x[n] * h[n]$. Write two expressions for $y[0]$, one (using the convolution sum directly) in terms of $x[n]$ and $h[n]$, and one (using the convolution property of Fourier transforms) in terms of $X(e^{j\omega})$ and $H(e^{j\omega})$. Then, by a judicious choice of $h[n]$, use these two expressions to derive Parseval’s relation—that is,

$$
\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.
$$

In a similar fashion, derive the following generalization of Parseval’s relation:

$$
\sum_{n=-\infty}^{+\infty} x[n]z^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Z^*(e^{j\omega})d\omega.
$$