1

1.1 a)
First we prove that given $A \rightarrow B$ and $B \rightarrow C$ we have $A \rightarrow C$, which we call the transitive property.

We have

\[(B \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))\] (A1)
\[B \rightarrow C\] (1)

(1), (A1) gives

\[A \rightarrow (B \rightarrow C),\] (2)
\[(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\] (A2)

(2), (A2) gives

\[(A \rightarrow B) \rightarrow (A \rightarrow C),\] (3)
\[A \rightarrow B\] (4)

(3), (4) reveal

\[A \rightarrow C,\]

which proves the transitive property. Now we have

\[(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)\] (A2)
\[(B \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))\] (A1)

According to the transitive property

\[(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))\] (5)

Now according to (A3)

\[((B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))) \rightarrow ((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)))\] (6)

Combining (5) and (6) we obtain the desired sentence:

\[(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)).\]
1.2 b) 

\[ A \rightarrow (B \rightarrow A) \quad (7) \]

\[ (B \rightarrow A) \rightarrow (\neg A \rightarrow \neg B) \quad (A4) \]

(7) and (A4) reveals

\[ A \rightarrow (\neg A \rightarrow \neg B). \quad (8) \]

1.3 c) 

(A3) gives

\[ (A \rightarrow (\neg A \rightarrow \neg B)) \rightarrow (\neg A \rightarrow (A \rightarrow \neg B)) \quad (9) \]

Combining (9) and 8 reveals

\[ \neg A \rightarrow (A \rightarrow \neg B). \]

2

Assume that there exist an onto map \( f \) from \( A \) to \( P(A) \). Consider the set

\[ Y = \{ x \in X : x \notin f(x) \}. \quad (1) \]

Because \( f \) is onto, there exist a \( y \in A \), such that \( Y = f(y) \). Now prove that both of the conditions \( y \in Y \) and \( y \notin Y \) end up to a contradiction.

3

First observe that instead of polynomials with rational coefficients, we can just consider the roots of polynomials with integer coefficients. Now for a polynomial \( P(x) = \sum_{i=0}^{n} a_i x^i \) with integer coefficients, define the function

\[ h(P) = n + \sum_{i=1}^{n} |a_i|. \quad (2) \]

For each \( n \in \mathbb{N} \), prove that the set of polynomials \( P \) with \( h(P) = n \) is finite, and conclude that the set of polynomials with integer coefficients in countable. Noting the fact that each polynomial has only finitely many roots, obtain the desired theorem.