Homework 6

Problems

1. Prove inequalities below, and give an example of a random variable such that inequality turns into an equality.

   (a) (Markov inequality) For a non-negative random variable $X$ and $a > 0$
   
   $$P[X \geq a] \leq \frac{E[X]}{a}$$

   (b) (Chebyshev inequality) For random variable $X$ with mean $\mu$ and Variance $\sigma^2$ and $a > 0$
   
   $$P[|X - \mu| \geq a] \leq \frac{\sigma^2}{a^2}$$

2. Define weak law of large numbers and prove it using Chebyshev inequality.

3. Define strong law of large numbers and explain the difference between weak and strong law of large numbers.

4. Explain central limit theorem and talk about its approximation error. Explain why the tail probability bound given by central limit theorem is not useful (Hint: it must something to do with the approximation error!).

5. Suppose that $X_1, ..., X_n$ are independent Poisson random variables with mean 1

   (a) Using Markov inequality give an upper bound for $P[\sum_{i=1}^{n} X_i > \frac{3}{2}n]$

   (b) Using central limit theorem give an approximation for $P[\sum_{i=1}^{n} X_i > \frac{3}{2}n]$

6. Suppose $X$ is a Cauchy random variable with below density function.

   $$f(x, \mu, \sigma) = \frac{1}{\pi \sigma (1 + (\frac{x-\mu}{\sigma})^2)}$$

   (a) Find mean, variance, median, and mode of the random variable $X$.

   (b) Suppose $X_1, X_2$ are Cauchy random variables with $\mu = 0$ and $\sigma = 1$. Find distribution of $X_1 + X_2$.

   (c) Suppose $X_1, ..., X_n$ are standard Cauchy random variables. Talk about convergence properties of population mean $\frac{1}{n} \sum_{i=1}^{n} X_i$. (You may write a code and simulate this problem!)

   (d) Explain why strong and week law of large numbers, and central limit theorem can’t be applied to this random variable.