Problem 1.

Consider the Fourier transform pair

\[ e^{-|t|} \overset{F}{\leftrightarrow} \frac{2}{1 + \omega^2} \]  

(a) Use the appropriate Fourier transform properties to find the Fourier transform of \( te^{-|t|} \)

(b) Use the result from part (a), along with the duality property, to determine the Fourier transform of

\[ \frac{4t}{(1 + t^2)^2} \]  

Problem 2.

Suppose \( g(t) = x(t) \cos t \) and the Fourier transform of the \( g(t) \) is

\[ G(j\omega) = \begin{cases} 1, & |\omega| \leq 2 \\ 0, & OW \end{cases} \]  

(a) Determine \( x(t) \).

(b) Specify the Fourier transform \( X_1(j\omega) \) of a signal \( x_1(t) \) such that

\[ g(t) = x_1(t) \cos \left( \frac{2}{3} t \right) \]
Problem 3. A causal and stable LTI system S has the frequency response
\[ H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega} \]  
(a) Determine a differential equation relating the input \( x(t) \) and output \( y(t) \) of S.
(b) Determine the impulse response \( h(t) \) of S.
(c) What is the output of S when the input is
\[ x(t) = e^{-4t}u(t) - te^{-4t}u(t) \]

Problem 4. Consider a causal LTI system with frequency response of
\[ H(j\omega) = \frac{1}{j\omega + 3} \]
For a particular input \( x(t) \) this system is observed to produce the output
\[ y(t) = e^{-3t}u(t) - e^{-4t}u(t) \]
Determine \( x(t) \)

Problem 5. Use properties of the Fourier transform to show by induction that the Fourier transform of
\[ x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at}u(t), a > 0 \]
is
\[ \frac{1}{(a + j\omega)^n} \]

Problem 6. Consider an LTI system whose response to the input
\[ x(t) = [e^{-t} + e^{-3t}]u(t) \]
is
\[ y(t) = [2e^{-t} - 2e^{-4t}]u(t) \]
(a) Find the frequency response of this system.
(b) Determine the system’s impulse response.
(c) Find the differential equation relating the input and the output of this system.

Problem 7. Suppose that a signal \( x(t) \) has Fourier transform \( X(j\omega) \). Now consider another signal \( g(t) \) whose shape is the same as the shape of \( X(j\omega) \); that is,
\[ g(t) = X(jt) \]
(a) Show that the Fourier transform \( G(j\omega) \) of \( g(t) \) has the same shape as \( 2\pi x(-t) \); that is, show that
\[
G(j\omega) = 2\pi x(-\omega)
\] (14)

(b) Using the fact that
\[
F\{\delta(t + B)\} = e^{jB\omega}
\] (15)
in conjunction with the result from part (a), show that
\[
F\{e^{jBt}\} = 2\pi\delta(\omega - B)
\] (16)

**Problem 8.** Consider the signals

\[
x(t) = u(t - 1) - 2u(t - 2) + u(t - 3)
\] (17)

and
\[
x(\tilde{t}) = \sum_{k=-\infty}^{\infty} x(t - kT)
\] (18)

where \( T > 0 \). Let \( a_k \) denote the Fourier series coefficients of \( x(\tilde{t}) \), and let \( X(j\omega) \) denote the Fourier transform of \( x(t) \).

(a) Determine a closed-form expression for \( X(j\omega) \).

(b) Determine an expression for the Fourier coefficients \( a_k \) and verify that
\[
a_k = \frac{1}{T}X\left(j\frac{2\pi k}{T}\right)
\]

**Problem 9.** Given the relationships

\[
y(t) = x(t) * h(t)
\] (19)

and
\[
g(t) = x(3t) * h(3t)
\] (20)

and given that \( x(t) \) has Fourier transform \( X(j\omega) \) and \( h(t) \) has Fourier transform \( H(j\omega) \), use Fourier transform properties to show that \( g(t) \) has the form
\[
g(t) = Ay(Bt)
\] (21)

Determine the values of A and B.