In the name of GOD.
Sharif University of Technology
Stochastic Processes  CE 695  Fall 2020  H.R. Rabiei

Homework 5 (UMVUE, Prediction, Filtering)

1. Use the Cramer-Rao Lower Bound to show that $\bar{X}$ is a UMVUE for $\theta$ based on a random sample of size $n$ from an $\text{exp}(\theta)$ distribution where:

$$f(x \mid \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad 0 < x$$

2. Suppose $X_1, \ldots, X_n$ are iid $\sim \text{Unif}[-\theta, \theta]$. Find a UMVUE of $\theta$.

3. Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta), \theta > 0$. Consider the following 3 estimators for $\theta$

$$T_1(X) = \frac{n+1}{n} X_{(n)}, T_2(X) = 2 \bar{X} \text{ and } T_3(X) = X_{(1)} + X_{(n)}$$

Show that all the estimators are unbiased for $\theta$. Among the three estimators, which one would you prefer and why?

4. Assume we want to estimate the random variable $X$ after observing $Y$:

A) Prove the following equation ($f : \mathbb{R}^n \to \mathbb{R}^m$ is an arbitrary function)

B) Using the above equation prove the minimum MSE estimator for $X$ is $E[X \mid Y]$

5. Assume we want to transfer a signal $X[n]$ which has auto-correlation

$$R_{XX}[m] = e^{-\alpha} \delta[m-1] + e^{-\beta} \delta[m-2] \quad (1)$$

if we observe the signal $Y[n]$ at the receiver knowing a white Gaussian noise $w[n]$ with variance $\sigma_w^2$ has been added to the transferred signal, how we can reconstruct signal $X[n]$ from $Y[n]$.

$$Y[n] = X[n] + w[n] \quad (2)$$

6. Suppose $X[n]$ is a WSS process with following auto-correlation:

$$R_{XX}[m] = \begin{cases} \frac{1}{m} & m > 0 \\ 2 & m = 0 \end{cases} \quad (3)$$

Design a filter that predict value of $X$ using previous three values.
7. Suppose \( \hat{x} = ay(0) + by(T) \) is a MSE Estimator for \( x = \int_0^T y(t) dt \). Prove that:

\[
a = b = \frac{\int_0^T R_{yy}(t)dt}{R_{yy}(0) + R_{yy}(T)}
\] (4)