Non-State Space Models

• Recall that non-state-space models like Reliability Block Diagrams and Fault Trees can easily be formulated and solved for system reliability, system availability and system MTTF

• Each component can have attached to it
  – A probability of failure
  – A failure rate
  – A distribution of time to failure
  – Steady-state and instantaneous unavailability

• Assuming
  – all the components failure events and repair events are independent of each other
  – simple logical relationships between system and its components

• Fast algorithms are available to solve large problems
State-space Models

• To model complex interactions between system components, we need State-space models.

• Example: Markov chains or more general State-space models.

• Many examples of dependencies among system components have been observed in practice and captured by State-space models.
State-Space Models (Contd.)

- Drawn as a directed graph
- State: each state represents a system condition
- Transitions between states indicates occurrence of events. Each transition can be labeled by
  - Probability: homogeneous Discrete-Time Markov Chain (DTMC); Chapter 7 of the text
  - Time-independent-Rate: homogeneous Continuous-Time Markov chain (CTMC); Chapter 8 of the text
  - Time-dependent rate: non-homogeneous CTMC; some examples in Chapter 8 of the text
  - Distribution function: Semi-Markov process (SMP); some examples in Chapter 7 & Chapter 8 of the text
  - Two distribution functions; Markov regenerative process (MRGP); not treated in the text
Continuous-time Markov Chains (CTMC)

• Discrete state space – Markov Chains

• For Continuous-time Markov chains (CTMCs) the time variable associated with the system evolution is continuous.

• In this chapter, we will mean a CTMC whenever we speak of a Markov model (chain).
Formal Definition

• A discrete-state continuous-time stochastic process \( \{X(t) | t \geq 0\} \) is called a Markov chain if for \( t_0 < t_1 < t_2 < \ldots < t_n < t \), the conditional pmf satisfies the following Markov property:

\[
P(X(t) = x | X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \ldots, X(t_0) = x_0)
= P(X(t) = x | X(t_n) = x_n)
\]

• A CTMC is characterized by state changes that can occur at any arbitrary time

• Index set is continuous.

• The state space is discrete
CTMC

• A CTMC can be completely described by:
  – Initial state probability vector for $X(t_0)$:
    \[ P(X(t_0) = k), \ k = 0, 1, 2, \ldots \]
  – Transition probability functions (over an interval)
    \[ p_{ij}(v, t) = P(X(t) = j | X(v) = i), \text{ for } 0 \leq v \leq t, \text{ and } i, j = 0, 1, 2, \ldots \]
    \[ p_{ij}(t, t) = \begin{cases} 
      1, & \text{if } i = j \\
      0, & \text{otherwise}
    \end{cases} \]
    and, \[ \sum_{j \in I} p_{ij}(v, t) = 1, \ \forall \ i; \ , 0 \leq v \leq t \]
Homogenous CTMC

• \( \{X(t)|t \geq 0\} \) is a (time-)homogenous CTMC iff 
  \[ p_{ij}(v, t) = p_{ij}(0, t - v), \]  abbreviated: \( p_{ij}(t - v) \)

• Thus, the conditional pmf can be written as:
  \[ p_{ij}(t) = P(X(t + v) = j|X(v) = i), \]  for any \( v \geq 0 \)

• State probabilities at a time \( t \) or pmf at time \( t \) is denoted by
  \[ \pi_j(t) = P(X(t) = j) \]

• Using the theorem of total probability
  \[ \pi_j(t) = \sum_{i \in I} P(X(t) = j|X(v) = i)P(X(v) = i) \]

If \( v = 0 \) in the above equation, we get
  \[ \pi_j(t) = \sum_{i \in I} p_{ij}(0, t)\pi_i(0) \]
CTMC Dynamics

Chapman-Kolmogorov Equation

\[ p_{ij}(v, t) = \sum_{k \in I} P(X(t) = j | X(u) = k, X(v) = i).P(X(u) = k | X(v) = i) \]

\[ p_{ij}(v, t) = \sum_{k \in I} p_{ik}(v, u)p_{kj}(u, t), 0 \leq v < u < t. \]

• Unlike the case of DTMC, the transition probabilities are functions of elapsed time and not of the number of elapsed steps

• The direct use of the this equation is difficult unlike the case of DTMC where we could anchor on one-step transition probabilities

• Hence the notion of rates of transitions which follows next

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Transition Rates

- Define the rates (probabilities per unit time):

  **net rate out of state** \( j \) **at time** \( t \):

  \[
  q_j(t) = -\frac{\partial}{\partial t} p_{jj}(v, t)|_{v=t} \\
  = \lim_{h \to 0} \frac{p_{jj}(t, t) - p_{jj}(t, t + h)}{h} = \lim_{h \to 0} \frac{1 - p_{jj}(t, t + h)}{h}
  \]

- **the rate from state** \( i \) **to state** \( j \) **at time** \( t \):

  \[
  q_{ij}(t) = \frac{\partial}{\partial t} p_{ij}(v, t)|_{v=t} \\
  = \lim_{h \to 0} \frac{p_{ij}(t, t) - p_{ij}(t, t + h)}{-h} = \lim_{h \to 0} \frac{p_{ij}(t, t + h)}{h}.
  \]
Kolmogorov Differential Equation

- The transition probabilities and transition rates are,

\[
p_{ij}(t, t + h) = q_{ij}(t) \cdot h + o(h), \quad i \neq j \quad (\lim_{h \to 0} \frac{o(h)}{h} = 0)
\]

\[
p_{jj}(t, t + h) = 1 - q_{j}(t) \cdot h + o(h), \quad i = j
\]

\[
p_{ij}(v, t + h) = \sum_k p_{ik}(v, u)p_{kj}(u, t + h) \Rightarrow
\]

\[
p_{ij}(v, t + h) - p_{ij}(v, t) = \sum_k p_{ik}(v, u)[p_{kj}(u, t + h) - p_{kj}(u, t)]
\]

- Dividing both sides by \( h \) and taking the limit,

\[
\frac{\partial p_{ij}(v, t)}{\partial t} = \left[ \sum_{k \neq j} p_{ik}(v, t)q_{kj}(t) \right] - p_{ij}(v, t)q_{j}(t), \quad 0 \leq v < t \text{ and } i, j \in I
\]

Kolmogorov’s forward equation
Kolmogorov Differential Equation (contd.)

- Kolmogorov’s backward equation,
  \[
  \frac{\partial p_{ij}(v,t)}{\partial v} = \left[ \sum_{k \neq i} p_{kj}(v,t)q_{ik}(v) \right] - p_{ij}(v,t)q_i(v).
  \]

- Writing these eqns. in the matrix form,\[
  \frac{\partial P(v,t)}{\partial t} = P(v,t)Q(t),
  \]
  \[
  \frac{\partial P(v,t)}{\partial v} = Q(v)P(v,t).
  \]

where, \( Q(t) = [q_{ij}(t)] \) and \( q_{ii}(t) = -q_i(t). \)

Note that, \( \sum_j q_{ij}(t) = 0 \) for all \( i. \)
Homogeneous CTMC

Specialize to HCTMC: all transition rates are time independent (Kolmogorov diff. equations):

\[ \frac{dp_{ij}(t)}{dt} = \left[ \sum_{k \neq j} p_{ik}(t)q_{kj} \right] - p_{ij}(t)q_j, \]
\[ \frac{d\pi_j(t)}{dt} = \sum_{i \neq j} \pi_i(t)q_{ij} - \pi_j(t)q_j. \]

• In the matrix form, (Matrix $Q$ is called the infinitesimal generator matrix (or simply Generator Matrix))

\[ \frac{dP(t)}{dt} = P(t)Q, \]
\[ \frac{d\pi(t)}{dt} = \pi(t)Q \]
Infinitesimal Generator Matrix

• Infinitesimal Generator Matrix $Q = \begin{bmatrix} q_{ij} \end{bmatrix}$
  For any off-diagonal element $q_{ij}, i \neq j$ is the transition rate from state $i$ to state $j$

• Characteristics of the generator matrix $Q$
  – Row sum equal to zero
  – Diagonal elements are all non-positive
  – Off-diagonal elements are all non-negative
NHCTMC vs. HCTMC

• In the non-homogeneous case, one or more transition rate is time dependent where time is measured from the beginning of system operation (global time) whereas in a semi Markov process rates are time dependent where time is measured from the entry into current state (local time)

• Markov property is satisfied at any time by an NHCTMC or HCTMC while the Markov property is satisfied by a semi Markov process only at epochs of entry (or exit) from a state

• Sojourn times in states of an HCTMC are exponentially distributed but this is not necessarily true for an NHCTMC or an SMP
Definitions

- A CTMC is said to be *irreducible* if every state can be reached from every other state, with a non-zero probability.
- A state is said to be *absorbing* if no other state can be reached from it with non-zero probability.
- Notion of *transient*, *recurrent non-null*, *recurrent null* are the same as in a DTMC. There is no notion of periodicity in a CTMC, however.
- Unless otherwise specified, when we say CTMC, we mean HCTMC
CTMC Steady-state Solution

- Steady state solution of CTMC obtained by solving the following balance equations ($\pi_j$ is the limiting value of $\pi_j(t)$):
  
  $$\pi Q = 0 \Rightarrow \sum_{i \neq j} \pi_i q_{ij} - \pi_j q_j = 0 \quad \text{subject to} \quad \sum_j \pi_j = 1$$

- Irreducible CTMCs with all states recurrent non-null will have steady-state $\{\pi_j\}$ values that are unique and independent of the initial probability vector. All states of a finite irreducible CTMC will be recurrent non-null.

- Measures of interest may be computed as weighted sum of steady state probabilities: $E[Z] = \sum_j r_j \pi_j$.
CTMC Measures

• For as a weighted sum of transient probabilities at time $t$:

$$E[Z(t)] = \sum_j r_j \pi_j(t)$$

• Expected accumulated quantities (over an interval of time $(0,t]$)

$$E[Y(t)] = E[\int_0^t Z(\tau) d\tau]$$

$$= \sum_j r_j \int_0^t \pi_j(\tau) d\tau = \sum_j r_j L_j(t),$$

where, $L_j(t) = \int_0^t \pi_j(\tau) d\tau$ and vector $L(t) = [L_j(t)]$

• $L_j(t)$ is the expected time spent in state $j$ during $(0,t]$ 

Finally then,

$$\frac{dL(t)}{dt} = L(t)Q + \pi(0)$$

(LTODE)
Different Analyses of HCTMC

• Steady State Analysis: Solving a linear system of equations:
  \[ \pi Q = 0 \text{ subject to } \sum_j \pi_j = 1 \]

• Transient Analysis: Solving a coupled system of linear first order ordinary differential equations, initial value problem with constant coefficients:
  \[
  \frac{dP(t)}{dt} = P(t)Q, \\
  \frac{d\pi(t)}{dt} = \pi(t)Q \quad \text{where } \pi(0) \text{ is given}
  \]

• Cumulative Transient Analysis: ODEIVP with a forcing function:
  \[ \frac{dL(t)}{dt} = L(t)Q + \pi(0) \]
Markov Reward Models (MRMs)
Markov Reward Models (MRMs)

- Continuous Time Markov Chains are useful models for performance as well as availability prediction.
- Extension of CTMC to Markov reward models make them even more useful.
- Attach a reward rate (or a weight) $r_i$ to state $i$ of CTMC. Let $Z(t)=r^X(t)$ be the instantaneous reward rate of CTMC at time $t$. 
3-State Markov Reward Model with Sample Path of $Y(t)$ Processes

![Sample Path of $Y(t)$ Processes](image)
Markov Reward Models (MRMs) (Continued)

• Expected instantaneous reward rate at time $t$:

$$E[Z(t)] = \sum_i r_i \pi_i(t)$$

Expected steady-state reward rate:

$$E[Z] = \sum_i r_i \pi_i$$