The WK-Recursive Pyramid: An Efficient Network Topology

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Abstract

In this paper, we present and evaluate a new topology for interconnection networks which we refer to as WK-pyramid. This network has a recursive definition quite similar to that of conventional pyramid networks, but is based on the WK-recursive mesh. The traditional pyramid, a desirable network topology, was employed as both software data structure and hardware architecture. However, the new network topology which we propose has, in addition to almost all the properties of the pyramid, a number of superior properties, such as expandability to any base size. We show that this hierarchical topology is of interesting topological characteristics, making it suitable for utilization as the base topology of large-scale multi-computer systems.

Keywords: Interconnection networks, Pyramids, WK-recursive, hierarchical networks, Hamiltonian cycle.

1. Introduction

Interconnection networks play a major role in the design of efficient parallel and distributed computing systems. Various topologies for static interconnection network have been proposed in the literature [1-9]. The tree, mesh, hypercube, k-ary n-cube, star graph, OTIS-Network and WK-recursive mesh are examples of common interconnection network topologies. Desirable properties of interconnection networks include symmetry, small node degree $d$, diameter $D$ and high connectivity, scalability, modularity and fault tolerance. In general, there is a tradeoff between hardware cost and message transmission time, respectively corresponding to node degree and the diameter of the network. Thus, the product of node degree and network diameter can be defined as a metric of network cost, indicating a combined measure of hardware complexity and worst-case message routing complexity.

The WK-recursive mesh, which was proposed in [1], is a class of recursively scalable networks, denoted by $WK(d,t)$, that is constructed hierarchically by grouping basic modules. Any $d$-node complete graph can serve as the basic modules. $WK(d,t)$ is a network of $t$ levels whose basic modules are $K_d$. We formally define this topology in Section 2.

Another useful network topology, which has been used as the base of both hardware architectures and software structures, is the pyramid. By exploiting the inherent hierarchy at each level, pyramids can handle various problems in graph theory, digital geometry, machine vision, and image processing [11]. Fault-tolerant properties of this network [12] make it also a promising network for reliable computing. Pyramids have therefore gained much attention in past studies [15-19] and numerous properties of them have been studied.

The present paper introduces a new topology, the definition of which is based on the WK-recursive mesh. This network preserves almost all of the desirable properties of the WK-recursive mesh and pyramid networks, and displays even better topological properties in some cases. For example, while being of low diameter, this network possesses a small number of edges. In addition, a pyramid network can easily be embedded onto this network.

The rest of the paper is organized as follows. Section 2 defines the WK-recursive mesh and the pyramid network, some useful notation and definitions which will be used in subsequent sections. In Section 3, the WK-pyramid topology is defined and some of its basic properties such as node degree, diameter and its addressing scheme have been studied. In Section 4, the Hamiltonian properties of the new pyramid are studied. Finally, Section 5 concludes this study.

2. The Pyramid and WK-Recursive Mesh

An interconnection network, which is based on the pyramid topology, consists of several layers of processing nodes. Each layer is arranged as a square mesh with half the dimensionality of the layer below it.
The highest layer, which has exactly one node, is called the apex (i.e. a 1×1 mesh).

**Definition 1.** An a×b mesh network, denoted $M_{a,b}$, consists of a set of nodes $V(M_{a,b}) = \{(x,y) \mid 1 \leq x \leq a, 1 \leq y \leq b\}$ where nodes $(x_i,y_i)$ and $(x_j,y_j)$ are connected by an edge if $|x_i-x_j| + |y_i-y_j| = 1$ [19].

**Definition 2.** A pyramid of n levels (layers), denoted $P_n$, consists of a set of nodes $V(P_n) = \{(k,x,y) \mid 0 \leq k \leq n, 1 \leq x,y \leq 2^k \}$. A node $(k,x,y) \in V(P_n)$ is said to be a node at level $k$. All the nodes in level $k$ form a $2^k \times 2^k$ mesh, $M_{2^k\times 2^k}$. Hence, there are totally $N = \sum_{k=0}^{n} 4^k = (4^{n+1}-1)/3$ nodes in a $P_n$. A node $(k,x,y)$ is connected, within the mesh at level $k$, to node $(k,x-1,y)$ if $x \geq 1$, to node $(k,x,y-1)$ if $y \geq 1$, to node $(k,x+1,y)$ if $x < 2^k$, and to node $(k,x,y+1)$ if $y < 2^k$, as neighbouring brother nodes (nodes at the same level). It is also connected to node $(k+1,x,2y)$, node $(k+1,2x,2y-1)$, node $(k+1,2x-1,2y)$, and node$(k+1,2x,2y)$, for $0 \leq k \leq n$, in level $k+1$, as child nodes, and to node $(k-1, x/2, y/2)$, in level $k-1$ as a father node [19]. The apex node in $P_n$ is the node with address (0, 1, 1).

**Definition 3.** An L-level WK-recursive mesh denoted as $WK_{(C,L)}$ consists of a set of nodes $V(WK_{(C,L)}) = \{a_1a_2...a_L \mid a_i \in \Omega \}$ for $1 \leq L \leq L_1$ which $\Omega = \{0,1,2,...,C-1\}$. The node with address scheme $A=(a_1a_2...a_L)$ is connected to $l$ all nodes with addresses $(a_1a_2...a_k)$ such that $k \in \Omega$, $k \neq a_i$, as brother nodes and 2) to node $(a_1a_2...a_ja_{j+1})$, if there exists one $j$ such that $1 \leq j \leq L-1$, $a_{j+1}=a_j=...=a_i$ and $a_j \neq a_{j+1}$; as cousin node. Notation $(x)^j$ denotes $j$ consecutive $x$’s. It is apparent that the node with address $a_la_1a_1...a_L$ has no cousin. We call such a node an extern node. Therefore, all $WK_{(C,L)}$ networks have exactly $C$ extern nodes. The degree of extern nodes in each $WK_{(C,L)}$ is $C-1$ and the degree of other nodes is $C$. As a result, the number of nodes in a $WK_{(C,L)}$ is equal to $|V(WK_{(C,L)})| = C^L$ [9]. The number of edges in a $WK_{(C,L)}$ is equal to $|E(WK_{(C,L)})| = C^{L-1}C-1$.

### 3. The WK- Pyramid networks

In this section, we present the proposed interconnection network topology. This topology eliminates some drawbacks of the conventional pyramid network, stemming from the fact that the connections within its layers form a mesh. Contrary to the pyramid, the connections within the layers of this network form a WK-recursive mesh, a sturdy topology. In view of this alteration, the base of this network can be of any size, not only 4 as is the case in the previous definition. Other properties of the network are also improved because of the superior property of the WK-recursive mesh compared to the mesh topology.

**Definition 4.** A WK-Pyramid network, denoted as $WKP_{(C,L)}$, consists of a set of nodes $V(WKP_{(C,L)}) = \{(k,a_1a_2...a_j) \mid 0 \leq k \leq L, 0 \leq a_i \leq C-1, 1 \leq L \leq k \text{ or } a_j=1 \}$. A node with addressing scheme $(k,a_1a_2...a_j)$ of the address determines the address of a node within the WK-recursive network at layer $k$. All the nodes in level $k$ form a $WK_{(C,L)}$ network. Hence, there exist totally of $N = \sum_{k=0}^{L} C^k = (C^{L+1}-1)/C-1$ nodes in a $WKP_{(C,L)}$. A node with address $(k,a_1a_2...a_j)$ is connected, within the WK-Recursive network at level $k$, to node $(k,a_1a_2...a_j)$ in $V(WKP_{(C,L)})$, for $1 \leq j \leq C$, as the neighbouring brother nodes, and connected to a node with address scheme $(k,a_1a_2...a_ja_{j+1})$, if there exists one $j$ such that $1 \leq j \leq L-1$, $a_{j+1}=a_j=...=a_i$ and $a_j \neq a_{j+1}$; as a cousin node (nodes at the same level). It is also connected to nodes $(k+1,a_1a_2...a_ja_{j+1})$, in level $k+1$, as a child node, and connected to node $(k-1,a_1a_2...a_ja_{j+1})$, in level $k-1$, as a father node. Figure 1 illustrates a WK-pyramid network, $WKP_{(4,2)}$.

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For example, in a $WKP_{(4,2)}$, the node with address $(2,1,1)$, is connected to nodes with addresses of form $(2,1,1)$, $(2,1,2)$, $(2,1,3)$, $(2,1,4)$ and $(1,1,1)$. As well, The
neighbour set of node \((2, (23))\) is \(\{(2, (21)), (2, (22)), (2, (24)), (2, (32)), (1, (2))\}\).

The node degree \(N_0\) is defined as the number of physical channels emanating from a node. This attribute is a measure of a node’s I/O complexity. For the \(WKP(C, L)\) node degree is independent of the network size. Each node in the intern layers has exactly \(2C+1\) neighbours, each node in layer 1 has just \(2C\) channels and each node in the bottom layer has \(C+1\) neighbours. The node at level 0 has just \(C\) neighbours, and each extern node at any layer has one neighbour less than the other nodes in the same layer.

The Diameter and average distance are commonly used to measure and to compare the static network performance of a system. The Diameter of a network is the maximum internodes distance i.e. is the maximum number of links that must be traversed to send a message to all nodes along a shortest path. The smaller the diameter of a network is, the less time it takes to send a message from one node to the farthest away node.

The diameter of \(WKP(C, L)\) with \(N = ((C^{i+1}-1)/(C-1))\) nodes is \(D = 2L-1 \approx 2\log C N\). So the diameter of the \(WKP(C, L)\) is \(O(\log N)\). Figure 2 (a) shows a comparison of network cost, which is defined as diameter \(\times\) node degree, for several different network topologies. Each curve of it represents a particular family of these networks. It is observed that the \(WKP(C, L)\) is of much less network cost than conventional mesh, hypercube, \(k\)-ary \(n\)-cube and WK-Recursive networks.

Average distance is the mean distance between all pairs of nodes in a network. A small average distance allows small communication latency, especially for distance-sensitive switching, such as store and forward [10]. But it is also crucial for distance-insensitive switching, such as wormhole switching, since short distance results in the use of fewer links and buffers, and therefore less communication contention and latency. We evaluate the average distance in different conventional topologies with that of WK-Pyramid networks with different numbers of levels by the use of simulation.

Figure 2 (b) shows a comparison of average distance in several different network topologies, including WK-pyramids of different sizes and of different numbers of levels. From these results, it is obvious that the \(WKP(C, L)\) is of smaller average distance than the other networks except hypercube networks.

4. Routing in the WK-pyramid network

Recall that every node in the \(WKP(C, L)\) has a node identifier address \((k(a_L, a_{L-1}…a_2…a_1))\) which \(0 \leq k \leq L\), \(0 \leq a_L \leq C-1\), \(1 \leq k \leq L\) or \(k = 0\) and \(a_1 = 1\).

We propose a routing algorithm for \(WKP(C, L)\) using the routing algorithm already proposed for WK-recursive networks in [1]. The routing algorithm in a \(WKP(C, L)\) is denoted by the self-routing algorithm, \(SR_{WKP(C, L)}(S,D)\), where \(S\) is the source node and \(D\) is the destination node. It exploits a deterministic routing in each layer to find the shortest path in these layers and so do the \(WKP(C, L)\) network. More precisely, the algorithm gets the message from the source node, deliver it to the every ancestors of the source node, then route it, with a routing algorithm we will state it later, through the related layer to the ancestor of the destination node, and finally delivers it to the destination node. The routing algorithm decides when to send the message to the higher layer. It is determined with comparison between the length of paths which was selected by the routing algorithm of each layers.

Let us first declare the routing algorithm in a WK-recursive mesh [1].

\(SR_{WKC(L)}(S,D)\) //the self-routing algorithm in WK-Recursive

Say \(S=(a_La_{L-1}…a_1)\) and \(D=(b_Lb_{L-1}…b_1)\) are consecutively the source and destination nodes.

If \(L=1\) \(\{SR=S \mid D\}\)
Else if \(a_L=b_L\) \(\{SR=SR_{WKC(L-1)}(S,D)\}\)
Else \(a_L\neq b_L\)
\{Say \(S'=a_Lb_Lb_{L-1}…b_1\) and \(D'=(b_La_La_{L-1}…a_1)\).
\(SR=SR_{WKC(L-1)}(S,S') \mid <S',D'>\)
\(\mid SR_{WKC(L-1)}(D,D')\}\}
\end of function

On the basis of the \(SR_{WKC(L)}(S,D)\) algorithm, we define the routing algorithm in WK-Pyramid network, named \(SR_{WKP(C, L)}(S,D)\). This algorithm will examine whether the level of the source and the destination nodes are identical or not. If they are equal, the algorithm recursively calls itself and finds the minimum path between two candidate paths, first the path which starts from the source node, would go to its father node and would go along a path between source’s father and destination’s father node which is returned by calling \(SR_{WKP(C, L)}(the\ father\ of\ S, the\ father\ of\ D)\), would go to the destination’s father and finally ends in the destination node. The second path is one that is located on layer \(k\) of pyramid and is returned by calling the \(SR_{WKP(C, L)}(S,D)\) algorithm on the WK-recursive network. When the levels of these nodes are not identical, the algorithm first checks whether the
source node is an ancestor of the destination or not. If so, the source node sends the message directly to its child and this is repeated until it reaches the destination node. In the other situation, the source node is not an ancestor of the destination node. To solve this case, two paths must be compared. First, we assume that the level of source node and that of the father of destination node is identical. So, the first path is between the source node and the father of the destination node, added up to the link between the father of destination node and itself. The second path is one that go through the source and its father, along a path between the source’s father node and the destination node which gain from calling SRWKP(C,L) algorithm recursively. A minimal path between these two paths is returned as a desired path. Therefore, the routing algorithm in WKP(C,L) network can be stated as follows:

SRWKP(C,L)(S,D) {
    Say S = (k, (a_k a_{k-1}...a_1)) and D = (k', (b_k b_{k-1}...b_1)), for 1 ≤ k, k' ≤ L, are consecutively the source and destination nodes.
    If (k = k') {
        S' = (k-1, (a_k a_{k-1}...a_1)) // the father of S
        D' = (k'-1, (b_k b_{k-1}...b_1))// the father of D
        SRWKP = min(<SS'|| SRWKP(C,L)(S',D')|| D'D>', SRWKP(C,L)(S,D))
    }
    Else if (k < k') {
        If (a_k a_{k-1}...a_1) = (b_k b_{k-1}...b_{k+2} b_{k+1}) // S is an ancestor of D
            S' = (k+1, (a_k a_{k-1}...a_1 b_{k+2} b_{k+1})) // the child of S that is an ancestor of D
            SRWKP = <SS'|| SRWKP(C,L)(S',D')>
        }
    }
}

5. Hamiltonian properties

In this section, we show that the WKP(C,L) network is Hamiltonian.

Definition 5. A Hamiltonian path between nodes u and v is a path that leads from u to v and visits each node of graph exactly once. A Hamiltonian cycle is a Hamiltonian path from one node to itself. A graph G is Hamiltonian if it contains a Hamiltonian cycle [19].

Theorem 1. Any WK-Recursive network is Hamiltonian-connected [7].

Theorem 2. Every WK-pyramid network of level L, WKP(C,L), contains a Hamiltonian path starting from any node x ∈ B={(0,1), (L,(k^L)) | 1 ≤ j, k ≤ C} and ending at any node y ∈ B - {x}.

Proof. With induction on L, the following paths determine the Hamiltonian path between two nodes (0, 1) and (1, j) or between (1, k) and (1, k + j), for some 1 ≤ k ≤ C and −k+1 ≤ j ≤ C-k, which proves the theorem when L=1.

\[
\begin{align*}
HP^0(0,1,1) & = <(0, 1) || (1, j + 1 \mod C) || (1, j + 2 \mod C) || \ldots || (1, j - 1 \mod C) || (1, j) > \\
HP^1(1,k,1) & = <(1, k) || (0, 1) || (1, k + 1 \mod C) || (1, k + 2 \mod C) || \ldots || (1, k + j - 1 \mod C) >
\end{align*}
\]

Figure 2. Comparison of WK-pyramid with other important networks; a) Network cost, b) Average inter-node distance.
(1, k + j + 1 mod C) || (1, k + j + 2 mod C) || ... || (1, k - 1 mod C) || (1, k + j mod C).

When L is greater than 1, two different cases must be considered:

1) When the source node address is (0, 1) and the destination node address is (L, (L+j)).

2) When neither the source nor the destination node has address of the form (0, 1), i.e. both of them have addresses of the form (L, ((L+j)+m)).

Case 1. We construct the Hamiltonian path, denoted with \( HP^L \), as follows:

\[
HP^L \left( 0, (L+j) \right) : = (0, 1) \parallel HP^{L-1} \left( (L+j), (L+j+1) \right) \parallel (L-1, (k+1) \mod C)^{L-1} \parallel (L, (k+1) \mod C)^{L} \parallel HP^{WK} \left( L, (L+k)^{L} \right) \parallel (L, (L+k)^{L})\.
\]

The \( HP^{WK} \) notation represents the Hamiltonian path in a WK-recursive network, the layer \( L \) of the WK-pyramid network, between nodes \( (L, (k+1) \mod C)^L \) and \( (L, k^L) \). As a result of Theorem 1, such a Hamiltonian path exists.

Case 2. Without loss of generality, let us assume that address \( x \) is \( (L, (j^L)) \) and address \( y \) is \( (L, (j+k)^L) \) for some \( 1 \leq j \leq C \) and \( j+k \leq k \leq C - j \). We only present proof of the case in which \( k \) and \( C \) are even; the other cases, in which either \( k \) or \( C \) is odd, can be determined in a similar manner. In this case, the following path determines a Hamiltonian path between these nodes.

\[
HP^L \left( (L+j)^L, (L+k)^L \right) = (0, 1) \parallel HP^{L-1} \left( (L+j)^L, (L+j+1)^L \right) \parallel (L, j^L+1(j+2)^L) \parallel (L, j^L+2(j+1)^L) \parallel (L, j^L+3j^L) \parallel (L, j^L+4(j+3)^L) \parallel \ldots \parallel (L, j^L+k) \parallel HP^{L-1} \left( (L+j)^L, (L+j+1)^L \right) \parallel (L, j^L+k+1(j+k+1)^L) \parallel (L, j^L+k+2(j+k+2)^L) \parallel \ldots \parallel (L, j^L+1) \parallel HP^{L-1} \left( (L+j)^L, (L+j+1)^L \right) \parallel (L, j^L+1) \parallel HP^{L-1} \left( (L+j)^L, (L+j+1)^L \right) \parallel (L, j^L+1) \parallel HP^{L-1} \left( (L+j)^L, (L+j+1)^L \right) \parallel (L, j^L+1) \parallel HP^{L-1} \left( (L+j)^L, (L+j+1)^L \right) \parallel (L, j^L+1) \parallel HP^{L-1} \left( (L+j)^L, (L+j+1)^L \right) \parallel (L, j^L+1) \parallel HP^{L-1} \left( (L+j)^L, (L+j+1)^L \right) \parallel (L, j^L+1) \parallel HP^{L-1} \left( (L+j)^L, (L+j+1)^L \right). \]

Considering the two above cases, the proof of our theorem is completed.

Figure 3 graphically displays the construction method used above for \( WKP_{(C,L)} \). This Figure displays the instance in which the address of the source node is \( (L, 0^L) \) and the address of destination node is \( (L, 4^L) \).

**Theorem 3.** Any \( WKP_{(C,L)} \) is Hamiltonian.

**Proof.** We use Theorem 2 to prove this theorem. In a \( WKP_{(C,L)} \) network, the following cycle (denoted as \( HC_{WKP_{(C,L)}} \)), is a Hamiltonian cycle:

\[
HC_{WKP_{(C,L)}} = (0, 1) \parallel (1, 1) \parallel HP^{L-1} \left( (L, 0^L), (L, 1^L) \right) \parallel (L, 2^L) \parallel HP^{L-1} \left( (L, 2^L), (L, 3^L) \right) \parallel (L, 3^L) \parallel HP^{L-1} \left( (L, 3^L), (L, 4^L) \right) \parallel (L, 4^L). \]

In this cycle, the notation \( HP^{L-1} \) represents a Hamiltonian path in a WK-recursive network which starts from the node \( x \) and ends at node \( y \).

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6. Conclusion

We have presented a new hierarchical interconnection network, called WK-pyramid network \( WKP_{(C,L)} \) for massively parallel computers. The issues presented the architecture of the \( WKP_{(C,L)} \), a routing algorithm, determining of the network cost, and discussion about some topological properties and finally a proof of existence of network’s Hamiltonian cycle and path.

Because of the desirable properties of this network, \( WKP_{(C,L)} \) is suitable for medium or large sized networks. It is shown that although the network degree of \( WKP_{(C,L)} \) is small, the average distance and diameter of this network is rather low, a property that it is desirable under wormhole and packet switching performance.

This topology can be utilized in many software applications that are traditionally implemented on the pyramid network, such as in graph theory, digital geometry, machine vision, and image processing [6]. This remains a subject for further exploration.
The conjecture of the authors is that the $WKP_{C,L}$ possesses two other important properties, the Hamiltonian-connectedness and Pancyclicity properties. Proving these properties is the aim of future work.

7. References