The RTCC-pyramid: A Versatile Pyramid network

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Abstract

We present and evaluate a new pyramid topology for interconnection networks, based on the Recursive Transpose-Connected Cycles (RTCC) network, which we refer to as the RTCC-pyramid. An L level RTCC-pyramid uses an RTCC network structure in each level as an alternative to the Mesh network used to construct the conventional mesh-pyramid network. We study some important properties of these topologies such as diameter and average distance. In addition, a routing algorithms based on the routing in RTCC network is proposed. We prove that this form of the pyramid topology is Hamiltonian-connected, i.e. a Hamiltonian path can be constructed from any node to any other node in the network. We conclude that, insight of most of the mentioned properties, the RTCC-pyramid is a more suitable topology to base both hardware architectures and software structures on, compared to the conventional pyramid topology.

1. Introduction

Interconnection networks are currently being used for many different applications, ranging from internal buses in VLSI circuits to wide area computer networks. An interconnection network can be modeled by a graph in which a processor is represented by a node, and a communication channel is represented by an edge between corresponding nodes. Various topologies for static interconnection network have been proposed in the literature [1-6]. The tree, mesh (especially the 2-dimensional mesh or $M_{2^n \times 2^n}$), hypercube, k-ary n-cube, OTIS-Network and WK-recursive mesh are examples of common interconnection network topologies. Desirable properties of interconnection networks include symmetry, small node degree $d$, diameter $D$, network cost (the product of node degree and network diameter, $dD$) and high connectivity, scalability, modularity, and fault tolerance.

The Recursive Transpose-Connected Cycles Network, which was proposed in [1], is a class of recursively scalable networks, denoted by $RTCC_{(C, L)}$ constructed hierarchically by grouping basic cycle modules. Any C-node cycle can serve as the basic modules. We define this topology in Section 2.

Another useful network topology which has been used as the base of both hardware architectures and software structures is the pyramid. By exploiting the inherent hierarchy at each level, pyramids structures can be efficiently exploited to handle various problems in graph theory, digital geometry, machine vision, and image processing [9-11]. Pyramids have therefore gained much attention in past studies and numerous properties of them have been studied.

The present paper introduces a new pyramid topology, the definition of a pyramid structure which is based on the RTCC Network. This network preserves almost all of the desirable properties of the mesh pyramid networks, and displays even better topological properties in some cases. In the following sections, we illustrate a number of these properties.

The rest of the paper is organized as follows. In Section 2, the RTCC is defined and the new pyramid network is proposed. We also delineate some useful notation and definitions which will be used in subsequent sections. As well, some of the basic properties of the RTCC-pyramid such as node degree, diameter and its addressing scheme are studied. Section 3, proof is presented of the fact that this network is Hamiltonian and Hamiltonian-connected. Finally, Section 4 concludes this study.

2. The RTCC and RTCC-pyramid network

In this section, we formally define the mesh-pyramid, the RTCC and RTCC-pyramid networks and derive some topological properties of the former. In this paper, we will apply the standard graph terminology.

2.1 Definition and Topological Properties

An interconnection network which is based on the pyramid topology consists of several layers of processing nodes. Each layer is arranged according to a certain conventional topology (a mesh network used in
formation of the traditional pyramid or the RTCC network presented in this paper) with half the dimensionality of the layer below it. The higher layer which has exactly one node is called the apex.

**Definition 1.** An axb mesh network, denoted *M*$_{a,b}$, consists of a set of nodes *V*(*M*$_{a,b}$) = \{(x,y) | 1≤x≤a, 1≤y≤b\} where nodes (x$_1$,y$_1$) and (x$_2$,y$_2$) are connected by an edge if |x$_1$-x$_2$|+|y$_1$-y$_2$|≤1 [12].

**Definition 2.** A mesh pyramid of *n* levels (or layers), denoted *P*$_n$, consists of a set of nodes *V*(*P*$_n$) = \{(k,x,y) | 0≤k≤n, 1≤x,y≤2$^k$\}. A node (k,x,y) ∈ *V*(*P*$_n$) is said to be a node at level *k*. All the nodes in level *k* form a 2$^{k}$×2$^{k}$ mesh. Hence, there are a total of \[N=\sum_{k=0}^{n} 4^k = (4^{n+1}-1)/3\] nodes in a *P*$_n$. A node (k, x, y) is connected, within the mesh at level *k*, to node (k,x-1,y) if x>1, to node (k,x,y-1) if y>1, to node (k,x+1,y) if x<2$^k$, and to node (k,x,y+1) if y<2$^k$, as neighbouring sister nodes (nodes at the same level). Each such node is also connected to node (k+1,x-2,y-1), node (k+1,x-1,y-1), and node (k+1,x-2,y), for 0≤k≤n, in level k+1, as child nodes, and to node (k-1,x,y-1), in level k-1 as a father node [12]. The apex node in *P*$_n$ is the node with address (0, 1, 1).

**Definition 3.** An L-level Recursive Transpose-Connected Cycles network denoted as RTCC(C, L), consists of a set of nodes *V*\{(C,L)|∈|C×L\} of the form (a$_L$a$_{L-1}$…a$_1$) where a$_i$ ∈ {0, 1…, C-1}. Therefore, the number of nodes of an RTCC(C, L) is equal to \[|RTCC(C, L)| = C \times 1 \times RTCC(C, L-1)| = C^L\] [1]. In the same manner, the number of edges of an RTCC(C, L) is equal to \[|RTCC(C, L)| = C \times 1 \times RTCC(C, L-1)| + C(C-1)/2 = (3C$^L$ - C)/2\]. We refer to the nodes with address of the form (a$_L$) as extern nodes. Therefore, the RTCC(C, L) networks have exactly *C* extern nodes. The address of the neighbors of a node with address (a$_L$a$_{L-1}$…a$_1$) are: (a$_L$a$_{L-1}$…(a$_1$+1)mod *C*) and (a$_L$a$_{L-1}$…(a$_1$-1)mod *C*) in the same lowest-level sub-graph, which we refer to as sister nodes, and (a$_L$a$_{L-1}$…a$_3$0a$_1$)\$^T\$ = a in the connection between the external nodes of level-L sub-graphs. We refer to this adjacent node as a cousin node. It is clear that the extern node with address (a$_L$)$^L$ is of no cousin. The degree of extern nodes of an RTCC(C, L) is 2 and the degree of all other nodes is 3.

**Definition 4.** An RTCC-pyramid network, denoted as P-RTCC(C,L), consists of a set of nodes *V*\{(k, a$_L$a$_{L-1}$…a$_1$) | 0≤k≤L, 0≤a$_i$≤C-1, 1≤i≤k or k = 0 and a$_1$=1\}. A node with addressing scheme (k, a$_L$a$_{L-1}$…a$_1$) is said to be a node at level *k*, e.g. the apex is at level 0. The part (a$_L$a$_{L-1}$…a$_1$) of the address determines the address of a node within the RTCC network at layer *k*.

All the nodes in level *k* form a RTCC(C,L) network. Hence, there exist a total of \[N = \sum_{k=0}^{L} C^k = (C^{L+1}-1)/C-1\] nodes in a P-RTCC(C,L). A node with address (k, a$_L$a$_{L-1}$…a$_1$) is connected, within the RTCC network at level *k* > 0, to node (k, a$_L$a$_{L-1}$…a$_1$(a$_1$±1)mod *C*), as the neighbouring sister nodes, and connected to a node with address schema (k, a$_L$a$_{L-1}$…a$_j$3a$_i$(a$_i$)\$^T\$), 1 ≤ j ≤ L-1 if there exists one *j* such that 1≤j≤L-1, a$_j$=a$_2$=…= a$_1$ and a$_i$ = a$_{j+1}$; as a cousin node (nodes at the same level). This node is also connected to nodes (k+1,a$_L$a$_{L-1}$…a$_1$), for 0≤s≤C, in level k+1, as a child node, and connected to node (k-1,a$_L$a$_{L-1}$…a$_1$), in level k-1, as a father node [12]. The apex node in *P*-RTCC is the node with address (0, 1, 1).

Figure 1 illustrates a RTCC-pyramid network, P-RTCC($4,2$).

![Figure 1. An P-RTCC(4, 2) network. We illustrate the apex node and the corner nodes of low-grade level in the network.](image-url)

Node degree (ND) is defined as the number of physical channels emanating from a node. For the RTCC-pyramid, node degree is independent of network size. Each node in the internal layers has exactly C+4 neighbours if it is not an extern node, and has C+3 neighbours if it is an extern node, each node in layer 1 has just C+3 channels and each non extern node in the bottom layer has 4 neighbours and each extern node in the last layer has 3 links. The node at level 0 has just C neighbours. Obviously each extern nodes at any layer has one neighbour less than the other nodes in the same layer. The number of edges in this network is \[|E(P-RTCC(C,L)| = 2.5 \times (C^{L+1}-C) / (C - 1 - CL)/2\].

The Diameter and average distance are commonly used to measure and to compare the static network performance of a system. The diameter of a network is the maximum number of links that must be traversed to send a message to all nodes along a shortest path. The smaller the diameter of a network is, the less time it takes to send a message from one node to the farthest away node.
The diameter of RTCC-pyramid with \( N = (C^{L-1} - 1)/(C - 1) \) nodes is \( D = 2L - 1 \approx 2 \log C \). The diameter of the \( P\text{-RTCC}(C,L) \) is thus \( O(\log N) \). Figure 2 shows a comparison of network cost, which is defined as diameter × node degree, for numerous different network topologies. Each curve represents a particular family of these networks. It is observed that the RTCC-pyramid is of much less network cost than more conventional mesh, hypercube and \( k\)-ary \( n\)-cube.

![Figure 2. A comparison of the network cost of popular networks.](image)

### 2.2 Message Routing in the \( P\text{-RTCC} \)

A recursive routing algorithm can be defined for routing a message from node \( S = (k,(a_1a_2…a_k)) \) to \( D = (k',(b_{k'}b_{k'-1}…b_1)) \), \( 0 \leq k, k' \leq L \), \( 0 \leq a_i, b_i \leq C-1 \), \( 1 \leq i \leq k \), in a \( P\text{-RTCC}(C,L) \) network.

We propose a routing algorithm for the RTCC-pyramid based on the routing algorithm proposed in [1] for RTCC networks. We denote this routing algorithm as \( R_\text{P-RTCC}(C,L)(S,D) \), where \( S \) and \( D \) shows the source and destination nodes. The algorithm exploits a deterministic routing in each layer to find the shortest path in these layers and so do in the \( P\text{-RTCC}(C,L) \).

Let us first revisit the routing algorithm presented for an RTCC network [1], denoted \( R_\text{GTCC}(C,L) \).

\( R_\text{RTCC}(C,L)(S,D) \) \( \text{if the self-routing algorithm in a RTCC network, it returns } R \text{ as a result route} \)

\[
\begin{cases}
\text{Say } S = (a_1a_2…a_k) \text{ and } D = (b_1b_2…b_{k'}) \text{ for } 1 \leq k, k' \leq L \\
\text{If } (k = k') \{
\text{If } (a_{k+1}a_{k+2}…a_{L}) = (b_{k+1}b_{k+2}…b_{L}) \} // \text{the child of } S \\
\text{Else if } (k < k') \{
\text{If } (a_{1}a_{2}…a_{k}) = (b_{1}b_{2}…b_{k}) \} // \text{is an ancestor of } D
\end{cases}
\]

\( P-R = \min \{ <SS'||P-R_\text{RTCC}(C,L-1)(S',D')||D'> : R_\text{RTCC}(C,L)(S,D) \} \)

The approach of this algorithm is as follows. When \( a_i = b_i \), the source and destination nodes belong to the same sub-graph. In this case, the same routing algorithm as that used to route the message from \((a_1a_2…a_k)\) to \((b_1b_2…b_{k'})\) in the \( RTCC(C,L) \) can be utilized. On the other hand, when \( a_i \neq b_i \), the message is first recursively routed from \( S \) to the node \( S' = (a_1b_1a_2b_2…a_kb_k) \), the cousin of \( S \). Finally from \( D' \), which resides within the destination sub-graph, the message is routed to \( D \). At the lowest level (where \( L = 1 \)), the same routing algorithm used for routing within a \( C\)-node cycle can be utilized.

On the basis of the \( R_\text{RTCC}(C,L) \) algorithm, we define the routing algorithm in an RTCC-pyramid network, named \( P-R_\text{RTCC}(C,L,S,D) \).

The main idea behind this algorithm is to identify the minimal path between the ancestors of the source node and the destination node. The routing algorithm decides when to stop sending the message to higher layers and route it along level-edges in the plane, by comparing the length of paths selected by the routing algorithm in layers by using \( R_\text{RTCC}(C,L) \) and recursively calling \( P-R_\text{RTCC}(C,L,S,D) \). The algorithm for routing in the \( P\text{-RTCC}(C,L) \) network can thus be stated as follows:

\[ P-R_\text{RTCC}(C,L,S,D) \] \( \text{if it returns } P-R \text{ as a resulting route} \)

\[
\begin{cases}
\text{Say } S = (k,(a_1a_2…a_k)) \text{ and } D = (k',(b_{k'}b_{k'-1}…b_1)) \text{ for } 1 \leq k, k' \leq L \\
\text{If } (k = k') \{
\text{If } (a_{k+1}a_{k+2}…a_{L}) = (b_{k+1}b_{k+2}…b_{L}) \} // \text{the father of } S \\
\text{Else if } (k < k') \{
\text{If } (a_{1}a_{2}…a_{k}) = (b_{1}b_{2}…b_{k}) \} // \text{is an ancestor of } D
\end{cases}
\]

\( P-R = \min \{ <SS'||P-R_\text{RTCC}(C,k-1)(S',D')||D'> : R_\text{RTCC}(C,k)(S,D') \} \)

Finally from \( D' \), which resides within the destination sub-graph, the message is routed to \( D \). At the lowest level (where \( L = 1 \)), the same routing algorithm used for routing within a \( C\)-node cycle can be utilized.
3. Hamiltonian paths and cycles in P-RTCC

The existence of a Hamiltonian path and cycle is an extremely important requirement for an interconnected network.

Definition 5: A Hamiltonian path between nodes u and v is a path that leads from u to v and traverses every node of G exactly once. A Hamiltonian cycle is a Hamiltonian path from one node to itself. A graph, G, is Hamiltonian if a Hamiltonian cycle can be found in the graph, and it is Hamiltonian-connected if a Hamiltonian path exists between every pair of nodes in G. We denote a Hamiltonian cycle and path in an P-RTCC as HC_{P-RTCC(C, L)} and HP_{P-RTCC(C, L)}, respectively.

Theorem 1. Any RTCC network is Hamiltonian [1].

Theorem 2. There exists a Hamiltonian path, in the RTCC_{C,L}, L ≥ 2, between any two extern nodes [1].

We denote the Hamiltonian path between two extern nodes in a RTCC_{C,L} network which starts from S and ends in D as HP_{RTCC(C, L)}(S, D).

Theorem 3. Every RTCC-pyramid network of level L, P-RTCC_{C,L}, contains a Hamiltonian path starting from any node x ∈ B = {0, 1}, L((a^k)) | 1 ≤ a ≤ C} and ending at any node y ∈ B - {x}.

Proof. B is a set of nodes that contains the apex node and extern nodes of level L of P-RTCC. We prove the theorem by induction on L. The base of induction corresponds to when L=1. We denote a Hamiltonian path in a P-RTCC_{C,1} which starts at S and ends in D as HP_{RTCC(C, 1)}(S, D). The following paths determine the Hamiltonian path between two nodes (0, (1)), the apex node, and (1, (j)) or between (1, (k)) and (1, (k+j)), for some 1 ≤ k ≤ C and k+j ≤ j ≤ C-k, which proves the theorem when L=1.

HP_{RTCC(C, 1)}((0, (1)), (1, (j))) = < 0, (1) || 1, (j+1modc) || 1, (j+2modc) || ... || 1, (j+1modc) >, HP_{RTCC(C, 1)}((1, (k)), (1, (k+j))) = < 1, (k) || 1, (k+1modc) || 1, (k+2modc) || ... || 1, (k+1modc) >.

To prove the theorem for L ≥ 1, we consider two different cases.

Case 1. Let S = (0, (1)) and the address form of D is (L, (k, l)). Then

HP_{RTCC(C, L)}((0, (1)), (L, (l-1modc))) = < 0, (1) || HP_{RTCC(C, L-1)}((0, (1)), (L-1, (l-1modc))) || HP_{RTCC(C, L-1)}((L-1, (l-1modc)), (L, (lmodc))) >.

Case 2. When neither the source nor the destination nodes have an address of the form (0, (1)), i.e. both of them have addresses of the form (L, (k, l)) and (L, (j + kmodc)) for some 1 ≤ k ≤ C and 1 ≤ j ≤ C - k. Four different situations must be solved. We deal only with the case in which both j and C are odd. The other situation can be handled in a similar manner. Thus,

HP_{RTCC(C, L)}((L, (k, l)), (L, (j + kmodc))) = < (L, (lmodc)) || HP_{RTCC(C, L-1)}((L, (lmodc)), (L-1, (l-1modc))) || HP_{RTCC(C, L-1)}((L-1, (l-1modc)), (L, (lmodc))) >.

Theorem 4. RTCC-pyramid network, P-RTCC_{C,L}, is Hamiltonian-connected.

Proof. We must show that P-RTCC_{C,L} contains a Hamiltonian path starting from an arbitrary node x ∈ V(P-RTCC) and ending at y ∈ V(P-RTCC) - {x}.

We consider three different cases to handle this problem, considered in the three following lemmas. All these cases correspond to when C is odd. Proof of the Hamiltonian-connectedness of the network when C is even can be obtained in a similar manner.

Lemma 1. Any RTCC-pyramid P-RTCC_{C,L} contains a Hamiltonian path starting from any node x ∈ V(P-RTCC) (except from apex node) and ending at the apex node in P-RTCC_{C,L}, (0, (1)).

Proof. With induction on the number of levels, L, for L=1, the lemma holds by Theorem 3. Otherwise, two cases can be considered.

Case 1. When the starting node, x, is the apex of any of the sub-pyramids P-RTCC_{C,L1}, i.e. x = (1, (j)), 1 ≤ j ≤ C. Without lose of generality, let us assume that x = (1, (j)). We construct the Hamiltonian path as follows:

HP_{RTCC(C,L)}((1, (j)), (1, (j))) = < (1, (j)) || HP_{RTCC(C,L-1)}((1, (j)), (1, (C))) || HP_{RTCC(C,L-1)}((1, (C)), (1, (C-1modc))) || ... || HP_{RTCC(C,L-1)}((1, (L-1modc)), (1, (L-1modc))) >.

Case 2. In this case, the node x is not an apex of any of the sub-pyramids. i.e. x ≠ (1, (j)). Without lose of generality, let us assume that x is of the form (l, (fa)) for some 1 ≤ l ≤ L. α indicates an arbitrary pattern such as α = w1w2w3w4w5 ... w11 and 1 ≤ w1 ≤ C. We build the Hamiltonian path like so:
There is a Hamiltonian path in the RTCC-pyramid initiating from a node of the form \((i, (ia))\) and finishing off at node \(y = (i', (iβ))\) for some \(1 \leq t, t' \leq L\), \(1 \leq i \leq C\) and \(α, β\) indicate an arbitrary arrangement of the form \(w_1, w_2, \ldots, w_k\) which \(1 \leq w_i \leq C\), i.e. the source and target nodes belong to the same sub-pyramid.

**Proof.** By induction on \(L\), assume that \(x\) and \(y\) belong to the first sub-pyramid, i.e. both of them have the address form of \(((i, (ia)))\) and \(((i', (iβ)))\). There are two cases that should be considered.

**Case 1.** Letting \(x ≠ (1, (I))\) and \(y ≠ (1, (I))\), as an induction hypothesis, assume that \(HP_{P-RTCC(C, L-1)}(x, y)\) embeds a Hamiltonian cycle in the first sub-pyramid of \(L-1\) level. We decompose this path to 3 disjoint parts as follow:

**Lemma 2.** The RTCC-pyramid \(P-RTCC_{(C, L)}\) includes a Hamiltonian path beginning from a node of the form \(x = (t, (ia))\) and ending up at node \(y = (t', (iβ))\). Other situations may be treated similarly. To solve this, we must regard 3 different cases.

**Case 1.** Let \(x = (1, (I))\) and \(y = (1, (2))\), i.e. both the source and destination nodes are apexes of sub-pyramids of level \(L-1\). The following is a desirable path in the RTCC-pyramid.

** Lemma 2.** The RTCC-pyramid \(P-RTCC_{(C, L)}\) includes a Hamiltonian path beginning from a node of the form \(x = (t, (ia))\) and ending up at node \(y = (t', (iβ))\). Other situations may be treated similarly. To solve this, we must regard 3 different cases.

**Case 2.** Let \(x = (1, (I))\) for \(1 < t \leq L\), i.e. \(x ≠ (1, (I))\) and \(y = (1, (2))\). Only one of the source and destination nodes is the apex of the sub-pyramid. The following path is a desirable path in the RTCC-pyramid.

**Case 3.** We assume \(x\) and \(y\) are of the form \(((i, (ia)))\) and \(((i', (2β)))\) for \(1 < t, t' \leq L\), i.e. \(x ≠ (1, (I))\) and \(y ≠ (1, (2))\). Neither the source nor the target node is the apex of the sub-pyramid. We thus build the Hamiltonian path as follows. The first and the last term must be built using lemma 1.

**Lemma 3.** There is a Hamiltonian path in the RTCC-pyramid initiating from a node of the form \(x = (t, (ia))\) and finishing off at node \(y = (t', (iβ))\) for some
We illustrate a Hamiltonian path obtained according to this method in Figure 3 in a $P$-$RTCC(5, L)$ between nodes $((1, 1), (1, 1))$ for $\zeta = 3$.

Using these lemmas, we conclude that every $L$ level RTCC-pyramid is Hamiltonian-connected, and theorem 3 is proven. □

4. Conclusions

Numerous network topologies have been proposed for multi-computer interconnection networks in the literature. In this paper, we introduced a new pyramid topology based on recursively defined cyclic networks (the RTCC network) instead of the mesh network. The low degree and diameter of the proposed network cause this network to be a good cost-effective candidate for interconnecting processing nodes in a multi-computer. Besides, some important properties such as Hamiltonian-connectivity of the RTCC-pyramid have been investigated.

Future work in this line includes a thorough performance evaluation of the RTCC-pyramid using extensive simulation experiments for different working conditions. In addition, an optimized broadcasting algorithm for this topology is under study.

5. References