Upper Bounding the Price of Anarchy in Atomic Splittable Selfish Routing

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Abstract. Selfish behavior of nodes of a network such as sensors of a geographically distributed sensor network, each of which owned and operated by a different stakeholder may lead to a game theoretic setting called "selfish routing". The fact that every node strictly aims at maximizing its own utility can cause degradations of social welfare. An issue of concern would be the quantitative measure of this inefficiency.
We study the price of anarchy in selfish routing games, a quantitative measure of inefficiency of worst equilibrium of the game imposed by noncooperative behavior of players. For the most of this paper, we consider atomic models of selfish routing in which the network is a multi-commodity flow network with multiple sources and sinks, and each player controls a considerable amount of the overall traffic. The price of anarchy for many variants of atomic splittable instances is not well understood, and upper bounding this parameter in presence of affine cost functions is the problems we tackle in this paper.

1 Introduction

A very popular question to address in real world communication networks is the way different elements of the network route information through the network to an intended destination. The need of communicating bits of information in a wide spread network is trivial, and it is not a surprise that so much effort has been put into finding better and more efficient methods of this communication. Several algorithms have been proposed for maximizing the overall performance of ad hoc networks, assuming that every node is willing to contribute to the social welfare. But this would not be the case when these intermediate elements act selfishly; they wish to incur as little cost of transmission (energy, delay, etc.) as possible, and to that end, the can calculate everything to find the perfect strategy. This is likely to happen is several settings. Consider a network of sensors installed in a specific site by different stakeholders. Each sensor is programmed in a way that solely tries to minimized its own cost in the network − it seeks to benefit the corresponding stakeholder. The lack of coordination between these self interested elements usually results in inefficiency.

Selfish behavior of network elements, such as sensors have been mostly dealt with in terms of energy consumption in theoretical computer science literature. Each selfish sensor wishes to do the least amount of work in the network, hence remaining alive as much as possible. Energy consumption can be critical when each sensor is equipped with a small battery. Yet it should be mentioned that satisfactory performance of such an element can not be measured just by the duration it keeps working. The quality of working matters too.

In our study of selfish behavior of network elements, we use a simplified measure of quality − the delay each node incurs. This is mostly due to the fact that adding more realistic, application-oriented parameters of quality in a communication would make our model so sophisticated that it would be too hard to perform a mathematical analysis on the model. More precisely, we look for a quantitative measure of inefficiency in terms of the overall communication delay time, caused by selfish behavior of participating nodes. The suitable notion that best models the outcome of this noncooperative behavior is Nash Equilibrium.

A vast portion of current literature on the inefficiency of equilibria is dedicated to selfish routing games in which the players are not allowed to route their flow of data or information through 2 or more different paths in the network (See *nonatomic selfish routing games* in [12], or *atomic unsplittable selfish routing games* in [1] for exmaple). However, this is not the case in many real world communication networks. Therefore, we choose to study the price of anarchy, the ration between the worst case equilibria of a game to the optimal flow (minimizing the overall delay of the network) in settings where players are allowed to freely split their amount of flow into different paths between their intended source and destination in the network. This model of selfish routing is quite naturally called *atomic splittable selfish routing* [13]. The model we use is described in details in the next subsection.

1.1 The Model

We use a slightly modified model of selfish routing that is well studied in the literature [1][4][6][7][8][9][11][13][14][12]. We consider "source routing" for the routing games discussed in this paper, in which each player chooses a complete path from the source to the destination for routing its traffic through. In general, there are two different classes of selfish routing games, namely *"non-atomic"* and *"atomic"* selfish routing of which, we only discuss the latter. The modifier *non-atomic* refers to multi-commodity networks in which each commodity represents a very large number of players, each controlling a negligible amount of the overall traffic. The *atomic selfish routing* differs with the former one in that in atomic settings, each commodity represents a player who controls a considerable amount of traffic.

More formally, we denote an atomic selfish routing game with a triple (G, d, c) where G represents a multi-commodity flow network (henceforward we refer to it as just a network for simplicity),and is given in the form of $G = (V, E)$ with vertex set V and directed edge set E. In addition, commodities are given in the form of the set $\{(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\}$ of source-sink vertices, where k is the number of players. Each player needs to route her traffic through different paths in the network originating in s_i and traveling to t_i. We denote the set of all these s_i-t_i paths by $\mathcal{P}_i$. We do not consider the networks in which $\mathcal{P}_i = \emptyset$ for some $i = 1, 2, \ldots, k$. We also denote the set of all paths in the network by $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$. In general, no capacity limit is imposed on the edges, and the graph G is allowed to contain parallel edges. The second coordinate in the triple, $d = (d_1, d_2, \ldots, d_k)$, is a vector of *demands*, a prescribed positive amount of the traffic that each player should route from the corresponding source to the sink. A flow f is said to be feasible for a vector d if for each $i \in \{1, 2, \ldots, k\}$, $\sum_{P \in \mathcal{P}_i} f_P = d_i$, where f_P is used to denote the amount of traffic going through the path P. Finally, c is the cost function for the network. More precisely, we denote the cost of each edge by $c_e : \mathrm{I\!R}^+ \rightarrow \mathrm{I\!R}^+$. We must consider some limitations for the choice of cost functions as we always assume that the cost function is nonnegative, continuous, and nondecreasing. The reason behind these assumptions is that they are all reasonable in applications where cost represents a quantity that increases with network congestion (e.g. delay time). Define the overall cost of a flow f incurred by a player i as

$$C^{(i)}(f) = \sum_{e \in E} c_e(f_e) \cdot f_e^{(i)} \tag{1}$$

$f_e^{(i)}$, the amount of traffic sent through the edge e by player i can be denoted by

$$f_e^{(i)} = \sum_{P \in \mathcal{P}_i : e \in P} f_P^{(i)} \tag{2}$$

in turn. Also let f_e be the total amount of traffic traveling through the edge e. It is immediate from out discussions that $f_e = \sum_{i=1}^{k} f_e^{(i)}$.

Remark. Note that the costs associated with the edges are in fact, cost per unit of traffic, meaning that if a player routes x amount of traffic on an edge e, she will be charged $x \times c_e(x)$ units of cost.

A feasible flow f in atomic splittable model in indexed by both players, and paths. In other words, the flow is a vector of the form $f = (f^{(1)}, f^{(2)}, \ldots, f^{(k)})$, in which each $f^{(i)}$ denotes the flow corresponding to player i, and in turn is a vector of the form $f^{(i)} = (f_{P_1}^{(i)}, f_{P_2}^{(i)}, \ldots, f_{P_{\|\mathcal{P}_i\|}}^{(i)})$, where $\|\mathcal{P}_i\|$ denotes the number of s_i-t_i paths. Note that we can also denote the total cost incurred by a player as a function of flow vectors of the individuals:

$$C^{(i)}(f) = C^{(i)}(f^{(1)}, f^{(2)}, \ldots, f^{(k)}). \tag{3}$$

Based on these basic notations, we give the definition of an equilibrium flow:

Definition 1. ***(Atomic Splittable Equilibrium Flow)*** *Let (G, d, c) be an atomic splittable instance, and f a feasible flow for this instance. f is an* equilibrium flow *for the network if, for every player $i \in \{1, 2, \ldots, k\}$ and every feasible flow $\hat{f}_i$,*

$$C^{(i)}(f^{(1)}, f^{(2)}, \ldots, f^{(i)}, \ldots, f^{(k)}) \leq C^{(i)}(f^{(1)}, f^{(2)}, \ldots, \hat{f}^{(i)}, \ldots, f^{(k)}).$$

1.2 Related Work

The nonatomic model of selfish routing was first formally defined by Wardrop [16]. Beckmann, McGuire, and Winsten [3] proved the existence and uniqueness of the equilibria in nonatomic models. Nonatomic selfish routing is immediately applicable for networks using source routing. In real world communication networks such as the Internet, another method of routing called distributed shortest-path routing is typically used instead of source routing. In this model, the user is not responsible for selecting a full path from the source to the receiver. For a discussion of this model see [9]. The price of anarchy in nonatomic selfish routing games was first studied by Roughgarden and Tardos [14]. For results on upper and lower bounds on the price of anarchy in this model, refer to [12], [11] and [14]. Finally, Roughgarden and Tardos suggested some way to reduce the price of anarchy in nonatomic settings [12].

Atomic selfish routing games were first considered by Rosenthal [10]. Rosenthal also proposed the concept of "congestion games" and also showed that equilibrium flows need not exist in weighted atomic instances − the instances in which the players do not control the same amount of traffic. The price of anarchy of atomic instances was first studied by Suri, Toth, and Zhou [15] in the context of the asymmetric scheduling games. For more results on the price of anarchy in atomic selfish routing, see Awerbuch et al. work in [1]. The notion of equilibrium used by the authors so far was *pure-strategy* Nash equilibrium. For detailed bounds on the price of anarchy using mixed-strategy Nash equilibrium, refer to [1] and [7]. The authors of [5] considered the price of anarchy and stability in different classes of asymmetric scheduling instances. See [2], [6], and [5] for results on the price of stability in atomic selfish routing games. Finally, several researchers have studied selfish routing in the atomic splittable model. Equilibrium flows in the atomic splittable model can behave in counterintuitive ways, and the price of anarchy in this model is not well understood. It was initially claimed that the upper bounds on the price of anarchy for nonatomic instances carry over to atomic splittable ones [8][13], but Cominetti, Correa, and Stier Moses [4] recently gave counterexamples to these claims in multicommodity networks. Obtaining tight bounds on the price of anarchy in this model remains an important open question.

1.3 Our Results

In this section, we first give a proof of existence of equilibrium in atomic splittable selfish routing instances with affine (i.e., linear) cost functions using a powerful method call *the potential function method*. Speaking generally, we propose a function on the outcome of the game (the flows) which reach its minimum if applied to an equilibrium flow. Then, we give a proof of the convexity of the function proposed and discuss the importance of convexity for any function to be a potential function for out model. Finally, we proceed to our mail result and propose an upper bound on the price of anarchy of such models.

2 Existence of Equilibrium

The proof presented in this paper is based on a powerful tool, namely the *Potential Function*. Roughly speaking, a potential function is defined on the outcomes of a selfish routing game in a way that a flow is an equilibrium one if, it is the global minimizer of potential function. More formally, we state this theorem:

Theorem 1. *Let (G, d, c) be an atomic splittable instance with k players and affine cost functions of the form $ax + b$. Such an instance always admits an equilibrium flow.*

Proof. In the proof of this theorem we assumed that cost functions associated with each edge are all affine. More precisely, we assume $c_e(fe) = a_e f_e + b_e$. Let Φ_{as} denote the potential function for atomic splittable instance, described as

$$\Phi_{as}(f) = \sum_{e \in E} \left(a_e \cdot \sum_{i=1}^{k} \sum_{j=i}^{k} f_e^{(i)} f_e^{(j)} + b_e \cdot \sum_{t=1}^{k} f_e^{(t)} \right). \tag{4}$$

First, we show that Φ_{as} is convex. Observe that for any real-values function defined on a vector space such as $\Phi_{as}(f)$ to be convex, a necessary and sufficient condition is that

$$\Phi_{as}(\frac{f + f'}{2}) \leq \frac{\Phi_{as}(f) + \Phi_{as}(f')}{2}.$$

So, we must prove that

$$\sum_{e \in E} \left[a_{e \in E} \cdot \sum_{i=1}^{k} \sum_{j=i}^{k} \left(\frac{f_e^{(i)} + {f'_e}^{(i)}}{2} \right) \left(\frac{f_e^{(j)} + {f'_e}^{(j)}}{2} \right) + b_e \cdot \sum_{t=1}^{k} \left(\frac{f_e^{(t)} + {f'_e}^{(t)}}{2} \right) \right] \leq$$

$$\frac{1}{2} \sum_{e \in E} \left(a_e \cdot \sum_{i=1}^{k} \sum_{j=i}^{k} f_e^{(i)} f_e^{(j)} + {f'_e}^{(i)} {f'_e}^{(j)} + b_e \cdot \sum_{t=1}^{k} f_e^{(t)} + {f'_e}^{(t)} \right).$$

Therefore,

$$\sum_{e \in E} \left[a_e \cdot \sum_{i=1}^{k} \sum_{j=i}^{k} \left(\frac{f_e^{(i)} f_e^{(j)} + {f'_e}^{(i)} {f'_e}^{(j)}}{2} - \frac{f_e^{(i)} f_e^{(j)} + f_e^{(i)} {f'_e}^{(j)} + {f'_e}^{(i)} f_e^{(j)} + {f'_e}^{(i)} {f'_e}^{(j)}}{4} \right) \right] \geq 0.$$

So,

$$\sum_{e \in E} \left[a_e \cdot \sum_{i=1}^{k} \sum_{j=i}^{k} \left(f_e^{(i)} f_e^{(j)} + {f'_e}^{(i)} {f'_e}^{(j)} - f_e^{(i)} {f'_e}^{(j)} - {f'_e}^{(i)} f_e^{(j)} \right) \right] \geq 0.$$

From which we can write:

$$\sum_{e \in E} \left[a_e \cdot \sum_{i=1}^{k} \sum_{j=i}^{k} (f_e^{(i)} - {f'_e}^{(i)})(f_e^{(j)} - {f'_e}^{(j)}) \right] \geq 0. \tag{5}$$

It can also easily be verified that:

$$\sum_{i=1}^{k}\sum_{j=i}^{k}(f_e^{(i)}-f_e'^{(i)})(f_e^{(j)}-f_e'^{(j)}) = \frac{1}{2}\left(\sum_{i=1}^{k}(f_e^{(i)}-f_e'^{(i)})\right)^2+\frac{1}{2}\sum_{i=1}^{k}\left((f_e^{(i)}-f_e'^{(i)})\right)^2. \tag{6}$$

By substituting (6) in (5), we get

$$\sum_{e\in E}\left[a_e\cdot\left(\frac{1}{2}\left(\sum_{i=1}^{k}f_e^{(i)}-f_e'^{(i)}\right)^2+\frac{1}{2}\sum_{i=1}^{k}\left(f_e^{(i)}-f_e'^{(i)}\right)^2\right)\right]\geq 0,$$

which is always true since a_e's are nonnegative. So, $\Phi_{as}(f)$ always admits a minimum value on its domain. We next show that every global minimizer of the potential function corresponds to an equilibrium flow of the network. For the sake of contradiction, assume that this does not happen. More formally, we choose a flow f such that it minimizes the potential function value, but we assume a player l can mitigate his cost by deviating to another split of the flow among the s_l-t_l paths. We call this new split $\hat{f}$. We can immediately write

$$C^{(l)}(f^{(1)},\ldots,f^{(l)},\ldots,f^{(k)})\geq C^{(l)}(f^{(1)},\ldots,\hat{f}^{(l)},\ldots,f^{(k)}).$$

As long as the other players do not deviate, for the flow $\hat{f}$ we have

$$\begin{cases}\hat{f}^{(i)}=f^{(i)}\\ \hat{f}_e^{(i)}=f_e^{(i)}\end{cases}, \text{ for } i\neq l.$$

Therefore:

$$\begin{aligned}0 &> C^{(l)}(f^{(1)},\ldots,\hat{f}^{(l)},\ldots,f^{(k)})-C^{(l)}(f^{(1)},\ldots,f^{(l)},\ldots,f^{(k)})\\ &=\sum_{e\in E}\left(c_e(\hat{f}_e)\cdot\hat{f}_e^{(l)}-c_e(f_e)\cdot f_e^{(l)}\right)\\ &=\sum_{e\in E}\left[\left(a_e\cdot\sum_{i=1}^{k}\hat{f}_e^{(i)}+b_e\right)\cdot\hat{f}_e^{(l)}-\left(a_e\cdot\sum_{i=1}^{k}f_e^{(i)}+b_e\right)\cdot f_e^{(l)}\right]\\ &=\sum_{e\in E}\left[a_e\cdot\sum_{i=1}^{k}\hat{f}_e^{(i)}\hat{f}_e^{(l)}+b_e\cdot\hat{f}_e^{(l)}-a_e\cdot\sum_{i=1}^{k}f_e^{(i)}f_e^{(l)}-b_e\cdot f_e^{(l)}\right]\end{aligned}$$

$$=\sum_{e\in E}\left[a_e\cdot\sum_{i\in\{1,\ldots,k\},i\neq l}(\hat{f}_e^{(l)}-f_e^{(l)})\hat{f}_e^{(i)}+a_e\cdot\left((\hat{f}_e^{(l)})^2-(f_e^{(l)})^2\right)+b_e\cdot(\hat{f}_e^{(l)}-f_e^{(l)})\right]. \tag{7}$$

We claim that equation (7) is exactly the amount of change that takes place in the value of potential function in case of deviation: The term $a_e\cdot\sum_{i\in\{1,\ldots,k\},i\neq l}(\hat{f}_e^{(l)}-f_e^{(l)})\hat{f}_e^{(i)}$, is the change in first term of (4) (the potential function) when exactly one of the variables i or j is equal to l. In case of $i, j \neq l$, no changes occur in neither the potential function and the equation (7). The second term of (7) is equal to the magnitude of change in first term of (4) when both i and j are equal to l. Finally, The last expression (with b_e coefficient) also appears in the amount of change in the potential function value if a player l deviates. Therefore potential function value decreases as player l deviated from the flow f to $\hat{f}$, which is in contradiction with the choice of f. Therefore, the global minimizer of Φ_{as} is in fact an equilibrium flow for the instance. As we have already showed the existence of a global minimizer for Φ_{as}, the proof is complete.

3 Upper Bounds on the Price of Anarchy

In this section, we obtain an upper bound on the price of anarchy of atomic splittable models based on the potential function. First, we prove the following two lemmas.

Lemma 1. *Let k be a positive integer and $\boldsymbol{V}$ be k-element vector of real numbers. Also let v_i denote the i^{th} element of the vector. The following is always true:*

$$\frac{k+1}{2}\sum_{i=1}^{k} v_i \geq \sqrt{\sum_{i=1}^{k} i \cdot v_i} \times \sqrt{\sum_{i=1}^{k}(k+1-i) \cdot v_i}.$$

Proof. define the real variable α as

$$\alpha = \sum_{i=1}^{k} i \cdot v_i,$$

and β as

$$\beta = \sum_{i=1}^{k}(k+1-i) \cdot v_i.$$

We know from basic algebra that the arithmetic mean is always greater than or equal to geometric mean. The arithmetic mean of α and β is

$$\frac{\alpha+\beta}{2} = \frac{1}{2}\left[\sum_{i=1}^{k} i \cdot v_i + \sum_{i=1}^{k}(k+1-i) \cdot v_i\right]$$

$$= \sum_{i=1}^{k} \frac{k+1-i+i}{2} \cdot v_i = \frac{k+1}{2}\sum_{i=1}^{k} v_i.$$

The geometric mean of α and β is

$$\sqrt{\alpha \cdot \beta} = \sqrt{\sum_{i=1}^{k} i \cdot v_i} \times \sqrt{\sum_{i=1}^{k}(k+1-i) \cdot v_i}.$$

We always have that $\frac{\alpha+\beta}{2} \geq \sqrt{\alpha \cdot \beta}$, hence the claim.

Lemma 2. *Let (G, d, c) be an atomic splittable instance and f, a feasible flow for the network. Also suppose that $\Phi_{as}(f)$ denotes the potential function for this instance. Then*

$$f_e c_e(f_e) \leq \frac{2k}{k+1} \cdot \left(a_e \sum_{i=1}^{k} \sum_{j=i}^{k} f_e^{(i)} f_e^{(j)} + b_e \sum_{t=1}^{k} f_e^{(t)} \right).$$

Proof. From the definition of f_e we have

$$f_e c_e(f_e) = \sum_{i=1}^{k} f_e^{(i)} \left(a_e \cdot \sum_{j=1}^{k} f_e^{(i)} + b_e \right)$$

$$= a_e \cdot \sum_{i=1}^{k} \sum_{j=1}^{k} f_e^{(i)} f_e^{(j)} + b_e \cdot \sum_{t=1}^{k} f_e^{(t)}.$$

We define Γ and Ψ in the following manner:

$$\Gamma = a_e \cdot \sum_{i=1}^{k} \sum_{j=1}^{k} f_e^{(i)} f_e^{(j)} + b_e \cdot \sum_{t=1}^{k} f_e^{(t)}, \tag{8}$$

$$\Psi = a_e \cdot \sum_{i=1}^{k} \sum_{j=i}^{k} f_e^{(i)} f_e^{(j)} + b_e \cdot \sum_{t=1}^{k} f_e^{(t)}. \tag{9}$$

Note that with above notation defined, the lemma we seek to prove would be of the form of

$$\Gamma \leq \frac{2k}{k+1} \Psi.$$

To prove so, we first compute $\Gamma - \Psi$.

$$\Gamma - \Psi = \left(a_e \cdot \sum_{i=1}^{k} \sum_{j=1}^{k} f_e^{(i) f_e^(j)} + b_e \cdot \sum_{t=1}^{k} f_e^{k} \right) - \left(a_e \cdot \sum_{i=1}^{k} \sum_{j=i}^{k} f_e^{(i)} f_e^{(j)} + b_e \cdot \sum_{t=1}^{k} f_e^{(t)} \right)$$

$$= a_e \cdot \sum_{i=1}^{k} \sum_{j=i+1}^{k} f_e^{(i)} f_e^{(j)}$$

$$= a_e \left(\sum_{i=1}^{k} \sum_{j=i}^{k} f_e^{(i)} f_e^{(j)} - \sum_{t=1}^{k} {f_e^{(t)}}^2 \right)$$

$$= \Psi - b_e \cdot \sum_{t=1}^{k} f_e^{(t)} - a_e \cdot \sum_{t=1}^{k} {f_e^{(t)}}^2. \tag{10}$$

On the other hand, using Cauchy-Schwarz inequality from linear algebra, we get

$$\sum_{i,j\in\{1,\ldots,k\},i\leq j} f_e^{(i)} f_e^{(j)} \leq \sqrt{\sum_{i,j\in\{1,\ldots,k\},i\leq j} {f_e^{(i)}}^2} \times \sqrt{\sum_{i,j\in\{1,\ldots,k\},i\leq j} {f_e^{(j)}}^2}. \quad (11)$$

It is easily validated that

$$\sum_{i,j\in\{1,\ldots,k\},i\leq j} {f_e^{(i)}}^2 = (k){f_e^{(1)}}^2 + (k-1){f_e^{(2)}}^2 + \ldots + (1){f_e^{(k)}}^2 = \sum_{i=1}^{k}(k+1-i){f_e^{(i)}}^2,$$

$$\sum_{i,j\in\{1,\ldots,k\},i\leq j} {f_e^{(j)}}^2 = (1){f_e^{(1)}}^2 + (2){f_e^{(2)}}^2 + \ldots + (k){f_e^{(k)}}^2 = \sum_{i=1}^{k} i \cdot {f_e^{(i)}}^2.$$

Therefore from these two equations together with (11) we can write:

$$\sum_{i,j\in\{1,\ldots,k\},i\leq j} f_e^{(i)} f_e^{(j)} \leq \sqrt{\sum_{i=1}^{k}(k+1-i){f_e^{(i)}}^2} \times \sqrt{\sum_{i=1}^{k} i \cdot {f_e^{(i)}}^2}. \quad (12)$$

Also from Lemma 1 it is straightforward to see

$$\sqrt{\sum_{i=1}^{k}(k+1-i){f_e^{(i)}}^2} \times \sqrt{\sum_{i=1}^{k} i \cdot {f_e^{(i)}}^2} \leq \frac{k+1}{2} \sum_{i=1}^{k} {f_e^{(i)}}^2. \quad (13)$$

Combining the two last results in (12) and (13) we get

$$a_e \cdot \sum_{i=1}^{k} {f_e^{(i)}}^2 \geq a_e \cdot \frac{2}{k+1} \sum_{i,j\in\{1,\ldots,k\},i\leq j} f_e^{(i)} f_e^{(j)}. \quad (14)$$

The following inequality is also trivial for $k \geq 1$

$$b_e \cdot \sum_{t=1}^{k} f_e^{(t)} \geq b_e \cdot \frac{2}{k+1} \sum_{t=1}^{k} f_e^{(t)}. \quad (15)$$

From (14) and (15),

$$\frac{2}{k+1}\left(a_e \cdot \sum_{i=1}^{k}\sum_{j=1}^{k} f_e^{(i)} f_e^{(j)} + b_e \cdot \sum_{t=1}^{k} f_e^{(t)}\right) \leq a_e \cdot \sum_{i=1}^{k} {f_e^{(i)}}^2 + b_e \cdot \sum_{t=1}^{k} f_e^{(t)}. \quad (16)$$

From substitution of (16) in (10) we arrive at

$$\Gamma - \Psi \leq \Psi - \frac{2}{k+1}\Psi = \frac{k-1}{k+1}\Psi.$$

Therefore

$$\Gamma \leq \Psi + \frac{k-1}{k+1}\Psi = \frac{2k}{k+1}\Psi.$$

In what follows, we propose our upper bound on the price of anarchy of atomic splittable selfish routing games with affine cost functions:

Proposition 1. *Let (G, d, c) be an atomic splittable instance with k players and affine cost function. Also let the flow f be the equilibrium and the flow f^* be the optimum flow that minimizes the overall cost. Then, $C(f) \leq \frac{2k}{k+1} C(f^*)$.*

Proof. From Lemma 2, and by summing the costs over the edges, the results would be

$$C(f) \leq \frac{2k}{k+1} \Phi_{as}(f) \leq \frac{2k}{k+1} \Phi_{as}(f^*) \leq \frac{2k}{k+1} C(f^*).$$

The first inequality is immediate from Lemma 2. The second inequality is because the equilibrium flow f is a global minimizer of the potential function. The third inequality is deduced from the fact that $C(f)$ always has some additional terms in comparison with $\Phi_{as}(f)$. More formally

$$\begin{aligned} \Phi_{as}(f) &= \sum_{e \in E} \left(a_e \left(\sum_{i,j \in \{1,\ldots,k\}, i \leq j} f_e^{(i)} f_e^{(j)} \right) + b_e \cdot \sum_{t=1}^{k} f_e^{(t)} \right) \\ &= C(f) - \sum_{e \in E} a_e \left(\sum_{i,j \in \{1,\ldots,k\}, i > j} f_e^{(i)} f_e^{(j)} \right). \end{aligned}$$

$\sum_{e \in E} a_e \left(\sum_{i,j \in \{1,\ldots,k\}, i > j} f_e^{(i)} f_e^{(j)} \right)$ is always greater than or equal to zero since for all edges $\in E$ and all players $i \in \{1, 2, \ldots, k\}$ we have $f_e^{(i)} \geq 0$, and also we choose a_e's to be nonnegative. This completes the proof.

References

[1] B. Awerbuch, Y. Azar, and L. Epstein, "The Price of Routing Unsplittable Flow", in *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC)*, 2005, pp. 57-66.

[2] E. Anshelevich, A. Dasgupa, J. Klienberg, . Tardos, T. Wexler, and T. Roughgarden, "The Price of Stability for Network Design with Fair Cost Allocation", in *Proceedings of 45th Annual Symposium on Foundation of Computer Science (FOCS)*, 2004, pp. 295-304.

[3] M. J. Beckmann, C. B. McGuire, and C. B. Winsten, "Studies in Economics of Transportation", *Yale University Press*, 1956.

[4] R. Cominetti, J. R. Correa, and N. E. Steir Moses, "Network Games with Atomic Players", in *Proceedings of the 33rd Annual International Colloquium in Automata, Languages, and Programming (ICALP)*, vol. 4051 of Lecture Notes in Computer Science, 2006, pp. 525-536.

[5] I. Caragiannis, M. Flammini, C. Kaklamanis, P.Kanellopoulos, and L. Moscardelli, "Tight Bounds for Selfish and Greedy Load Balancing", in *Proceedings of the 33rd Annual International Collequium in Automata, Languages, and Programming (ICALP)*, vol. 4051 of Lecture Notes in Computer Science, 2006, pp. 311-322.

[6] G. Christodoulou and E. Koutsoupias, "On the Price of Anarchy and Stability of Correlated Equilibria of Linear Congestion Games", in *Proceedings of 12th Annual European Symposium on Algorithms (ESA)*, 1005, pp. 59-70.
[7] G. Christodoulou and E. Koutsoupias, "The Price of Anarchy of Finite Congestion Games", in *Proceedings of 37th Annual ASM Symposium on Theory of Computing (STOC)*, 1005, pp. 67-73.
[8] J. R. Correa, A. S. Schulz, and N. E. Steir Moses, "On the Inefficiency of Equilibria in Congestion Games", in *Proceedings of the 11th Annual Conference on Integer Programming and Combinatorial Optimization (IPCO)*, 2005, pp. 142-151.
[9] L. Qui, Y. R. Yang, Y. Zhang, and S. Shenker, "On Selfish Routing in Internet-like Environments", in *Proceedings of ACM Conference on Communication, Protocols, and Applications (SIGCOMM)*, 2003, pp. 151-162.
[10] R. W. Rosenthal, "The Network Equilibrium Problem in Integers", in *Networks*, vol. 3(1), 1973, pp. 53-59.
[11] T. Roughgarden, "The Price of Anarchy is Independent of the Network Topology", in *Journal of Computer and System Sciences (JCSS)*, vol. 67(2), 2003, pp. 341-364.
[12] T. Roughgarden and . Tardos, "Bounding the Inefficiency of Equilibria in Nonatomic Congestion Games", in *Games and Economic Behavior*, vol. 47, 2004, pp. 389-403.
[13] T. Roughgarden, "Selfish Routing and the Price of Anarchy", *MIT Press*, 2005.
[14] T. Roughgarden, and . Tardos, "How bad is Selfish Routing?", in *Journal of the ACM*, vol. 49(2), 2002, pp. 236-259.
[15] S. Suri, C. Tth, and Y. Zhou, "Selfih Load Balancing and Atomic Congestion Games", in *Algorithmica*, vol. 47(1), 2007, pp. 79-96.
[16] J. G. Wardrop, "Some Theoretical Aspects of Road Traffic Research", in *Proceedings of the Institute of Civil Engineering*, Pt. II, vol. 1, 1952, pp. 325-378.

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