A Better Bound for the Cop Number of General Graphs

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Abstract: In this note, we prove that the cop number of any \( n \)-vertex graph \( G \), denoted by \( c(G) \), is at most \( O(\sqrt{n \log n}) \). Meyniel conjectured \( c(G) = O(\sqrt{n}) \). It appears that the best previously known sublinear upper-bound is due to Frankl, who proved \( c(G) \leq (1 + o(1)) \frac{\log n}{\log \log n} \).

Keywords: graph; caterpillar; cops and robbers; search number

Let \( G = (V, E) \) be an undirected, finite, simple, and connected graph, and \( n \) be the number of vertices of \( G \). The 2-player game of cops and robbers on \( G \) is as follows: at time 0, the first player, \( C \), places \( k \) cops on the vertices of \( G \). Next, at time 1, the second player, \( R \), places a robber on a vertex of \( G \). At times \( 2t, t \geq 1 \), \( C \) chooses a subset of the \( k \) cops and moves each selected cop to an adjacent vertex. At times \( 2t + 1, t \geq 1 \), \( R \) either moves the robber to an adjacent vertex, or does nothing. \( C \) wins if at some time a cop is in the same vertex that the robber is in; otherwise, if \( R \) can avoid this situation for ever, \( R \) wins. This game is studied by Aigner and Fromme [1], Quilliot [6], Andreae [3], Frankl [5], Berarducci and Intrigila [4], and several other people. We refer the reader to a recent survey by Alspach on some variations of the game of cops and robbers [2]. The minimum value of \( k \) for which \( C \) has a winning strategy is called the cop number of \( G \) and is denoted by \( c(G) \). The cop number of disconnected graphs is defined as the maximum cop number among their connected components.
There is a little known about the cop number of general graphs. Meyniel conjectured $c(G) = O(\sqrt{n})$ [5]. However, to the best of our knowledge, Frankl is the only one that obtained a sublinear upper bound for the cop number of general graphs. He proved $c(G) \leq (1 + o(1))\frac{\log n}{\log \log n}$ [5]. We improve this bound to $O(\frac{n \log n}{\log \log n})$.

In order to prove our result, we need the following definition: a minimum distance caterpillar between $u \in V$ and $v \in V$ is a subgraph $H$ of $G$ such that (1) $H$ is a tree, (2) the path $P$ between $u$ and $v$ in $H$ is a shortest path between $u$ and $v$ in $G$, and (3) every vertex of $H$ has an edge to a vertex in $P$. Figure 1 illustrates an example of a minimum distance caterpillar in a graph.

Aigner and Fromme [1] proved that if $P$ is a shortest path between $u$ and $v$ in $G$, then one cop can control $P$ which means the cop can move only in the vertices of $P$ in such a way that, after a finite number of moves, if the robber moves onto $P$ at any time $t$, he will be caught at time $t+1$. We allow a slightly larger number of cops, and generalize their result on $P$ to any minimum distance caterpillar.

**Theorem 1.** Every minimum distance caterpillar $H$ of $G$ can be controlled by 5 cops.

**Proof.** Assume that $H$ is a minimum distance caterpillar between $u$ and $v$, and $P = \{p_1 = u, p_2, \ldots, p_\ell = v\}$ is the simple path that connects $u$ and $v$ in $H$. Then, by the above-mentioned result of Aigner and Fromme, $P$ can be controlled by one cop $C_0$. Now, we put two cops $C_1$ and $C_2$ in front of $C_0$ in $P$ and two cops $C_{-1}$ and $C_{-2}$ behind $C_0$ in $P$; we make sure that at any time if $C_0$ is in $p_i$, $C_j$ is in $p_{i+j}$, for all $j \in \{-2, -1, 1, 2\}$ (we define $p_{-1} = p_0 = u$ and $p_{\ell+1} = p_{\ell+2} = v$). It is easy to check that we can move $C_j$s according to the game rules such that the mentioned property is satisfied.

If the robber moves onto $P$ at time $t$, then $C_0$ will catch the robber at time $t+1$. Consider the case that the robber moves onto a vertex of $H$ that is not in $P$, say $x$, at time $t$. Let $N_G(x)$ denote the set of all neighbors of a vertex $v$ in a graph $G$, and let $p_t$ be the neighbor of $x$ that is in $P$, that is, the vertex in $N_H(x) \cap P$. $C_0$ must be in one of the vertices of $\{p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2}\}$ in time $t$; otherwise, the robber could move to $p_t$ at time $t+2$ and $C_0$ would be unable to catch the robber at time $t+3$. Therefore, since $C_0$ is in one of the vertices of $\{p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2}\}$, one of the cops $C_{-2}, C_{-1}, C_0, C_1, C_2$, say $C_j$, is in $p_t$ at time $t$. So, $C_j$ can capture $x$. 

![Figure 1](https://example.com/figure1.png)
the robber at time \( t + 1 \). Thus, after a finite number of steps, if the robber moves to any vertex of \( H \) at time \( t \), he will be caught at time \( t + 1 \).

If \( C \) can find large enough minimum distance caterpillars in graphs, then \( C \) can control the whole graph with a small number of cops, because each minimum distance caterpillar can be controlled by only a constant number of cops.

**Theorem 2.** Every \( n \)-vertex connected graph has a minimum distance caterpillar of order at least \( \lg n \).

**Proof.** Suppose \( r \) is an arbitrary vertex in \( G \) and \( T \) is a breadth-first search tree of \( G \), rooted at \( r \). Any root-to-leaf path in \( T \) is a shortest path in \( G \). Thus, it is enough to show that \( T \) has a root-to-leaf path \( P \) such that the order of \( P \) plus the number of vertices that are adjacent to a vertex in \( P \) is at least \( \lg n \). For the rest of the proof, we ignore \( G \) and focus only on \( T \). For each path \( P \), we denote the number of vertices that are in \( P \) or are adjacent to a vertex in \( P \) by \( f(P) \).

We use induction to prove that every tree of order \( k \) has a root-to-leaf path \( P \) such that \( f(P) \geq \lg k \). This is trivially true for \( k = 1 \) and \( k = 2 \). Assume that the induction hypothesis is true for all \( k \leq n - 1 \). Consider a tree \( T \) of order \( k = n \). Let \( r \) be the root of \( T \). Then, there is a vertex \( s \) in \( N_T(r) \), such that the subtree of \( T \) rooted at \( s \) has at least \( \frac{n-1}{|N_T(r)|} \leq n - 1 \) vertices. Therefore, the subtree of \( T \) rooted at \( s \) has a root-to-leaf path \( P_s \) such that \( f(P_s) \geq \lg \left( \frac{n-1}{|N_T(r)|} \right) \). We add \( r \) to \( P_s \) to build a root-to-leaf path \( P_r \) in \( T \). Since \( r \) has \( N_T(r) - 1 \) neighbors that are not in \( P_s \) and are not adjacent to any vertex in \( P_s \), we have \( f(P_r) \geq f(P_s) + 1 + |N_T(r)| - 1 \geq \lg(n - 1) - \lg(|N_T(r)|) + |N_T(r)| \geq \lg n \).

Note that by analyzing the recursion at the end of the above-mentioned proof, we can get a slightly better bound; however, we ignore constant factors in this work.

**Corollary 3.** The cop number of any \( n \)-vertex graph is in \( O\left( \frac{n}{\lg n} \right) \).

**Proof.** Using the method described in the proof of Theorem 2, \( C \) can find a minimum distance caterpillar \( H \) in \( G \) of order at least \( \lg n \). Then, \( C \) can control \( H \) with 5 cops. Thus, after a finite number of steps, the robber is forced to avoid entering \( H \). After this, \( C \) can continue the game on the remaining graph. If eliminating the vertices of \( H \) from \( G \) makes \( G \) disconnected, \( C \) can continue the game in the connected component in which the robber is. Thus, if we denote the graph obtained by removing the vertices of \( H \) from \( G \) by \( G - H \), we have \( c(G) \leq c(G - H) + 5 \).

Let \( g(n) \) be the maximum value of \( c(G) \) among all graphs \( G \) with at most \( n \) vertices. We have \( g(n) \leq g(n - \lg n) + 5 \), and thus, \( g(n) \leq g\left( \frac{n}{2} \right) + \frac{5n}{\lg 2} \). Using a simple induction, we can prove that \( g(n) \) is at most \( \frac{20n}{\lg n} \).

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