Virtual Sensors: Model based Output Prediction

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Abstract

The problem of predicting the output of a system at regular instants from irregularly sampled signals is addressed. A general predictor based on a Kalman filter that can be applied for arbitrary measurement conditions is described. The particularization of this predictor for different sampling situations is then derived. Finally, some application examples are developed (binary sensors, sensor fusion and missing data) illustrating the main ideas.

1 Introduction

Consider a process with one control input and several measurable signals (states or outputs) that can be measured with different rates and in different time instants. Some of them could be measured with redundant sensors of different precision and sampling rates. Assume that the control system needs the values of some of these variables (or other ones that are not directly measurable but are a function of the measurable ones) at a fixed frequency on predefined time instants (when the control action is to be calculated and applied). In this scenario a virtual sensor is defined as the intelligent device that measures the available outputs or states at arbitrary instants and predicts the values of the desired output signals at the instants and frequency needed by the control system.

Several approaches can be used for this prediction. One possibility is to interpolate or extrapolate the measurements with any interpolation method. This idea could result in a computationally simple predictor. However, if the frequency of the measurements is not high enough the predictions could have a large error. In these situations the predictions can be significantly improved if a model of the process is used and the inputs as well as the output measurements are taken into account in the predictor algorithm. For this purpose, the control system must communicate the control actions to the virtual sensor. The model of the process can be identified off line or can be estimated online by an adaptive algorithm running on the virtual sensor.

In [1] the problem of identification when the output is measured synchronously with the input update but some measurements are missing is addressed. In that paper, a very simple predictor based on an input-output model of the process is used to predict the unmeasured outputs. The approach consists of substituting the predicted output on the difference equation by the measured one when this is available. In [5] and [6] the same problem is studied but a Kalman filter is used to predict the unmeasured outputs.

The study of output predictors for systems with irregular output measurements and missing data has been addressed by the authors in previous works ([2] and [7]), where different possible predictors have been analyzed, not only for the synchronous sampling case, but also for the asynchronous one. In [8] and [3] the stability of some of these predictors based on input-output models is analyzed.

In this paper a more general case of unmeasured output prediction is studied, because several signals that are related to the output are assumed to be measured at different instants (the output may be one of them, but not necessarily).

2 Problem Statement

The figure 1 represents a virtual sensor. It consists of several sensors that measure different variables at different sampling rates (possibly time varying). Some of these sensors can measure the same variable (with different rate and precision). The predictor takes as inputs these irregularly sampled values and the sequence of control actions at a fixed rate \(T\) (defined by the control system) and predicts the desired output variable at the same fixed rate \(T\) needed by the controller. In this paper a general predictor based on the process model is described and its application to different measurement patterns is analyzed (synchronous or asynchronous with the input update, regular or random sampling times).

The process is assumed to be a continuous linear time-invariant system described by the equations

\[
\begin{align*}
\dot{x}(t) &= A_c x(t) + B_c u(t) + \omega(t) \\
y(t) &= C_y x(t) \\
z(t) &= C_z x(t) + v(t)
\end{align*}
\]  
(1)
where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R} \) is the control input, \( y \in \mathbb{R} \) is the output signal to be controlled and \( z \in \mathbb{R}^m \) is the vector of measured signals. The signals \( \omega(t) \) and \( v(t) \) are possible input disturbances and measurement noise respectively. If the input is updated at a fixed period \( T \) through a ZOH \( (u(t) = u(kT), \ kT \leq t < (k+1)T) \), there exists a discrete time equivalent model that relates the values of the discrete input sequence and the values of the states and outputs at the input updating instants.

The control system needs the values of \( y(kT) \), but it is assumed that they are not available. Instead, the values of some of the elements of the vector \( z(t) \) are measured at different sampling instants. The predictor is a dynamic system whose inputs are the control action at the control rate \( T \) and the irregularly sampled values of \( z(t) \) and whose output is the predicted value of \( y(kT) \). For the general case where the measurements are assumed to be taken at arbitrary time instants, a Kalman filter can be used to estimate the state and then the output at the input updating instants. The next section describes this in detail.

### 3 Kalman filter based output prediction

#### 3.1 General case: arbitrary measurement instants

Assume that the disturbances \( \omega(t) \) and \( v(t) \) in model (1) are white noise with variance-covariance matrices \( W \) and \( V \) respectively \( (E(\omega(t)\omega^T(\tau)) = W\delta(t-\tau)). \)

Assume that some of the elements of vector \( z(t) \) are measured at arbitrary instants \( t_i, \ i \in \mathcal{N} \) (at least one element of \( z(t) \) is measured at instant \( t_i \)). Let us define the \( m_i \times m \) matrix \( M_i \) formed with the rows of an identity matrix that correspond to the position of the elements of \( z(t) \) measured at instant \( t_i \) (thus \( m_i \) is the number of variables measured at instant \( t_i \)).

The state estimator that minimizes the error variance (Kalman filter) is defined by the equations

\[
\hat{x}(t) = A_c\hat{x}(t) + B_cu(t)
\]

that are run along the interval between measurements, plus the equations

\[
\dot{\hat{x}}(t_i^+) = \hat{x}(t_i^-) + L(t_i) \left[ M_i z(t_i) - M_i C_2 \hat{x}(t_i^-) \right]
\]

\[
L(t_i) = \hat{Q}(t_i^-) (M_i C_2)^T \left[ M_i C_2 \hat{Q}(t_i^-) (M_i C_2)^T + M_i V M_i^T \right]^{-1}
\]

\[
\hat{Q}(t_i^+) = [I - L(t_i) M_i C_2] \hat{Q}(t_i^-)
\]

that are applied at the measurement instants. The continuous time differential equations (2) and (3) can be easily integrated along the inter measurement intervals. Defining the time between two consecutive measurements \( \tau_t = t_i - t_{i-1} \), the resulting equations are

\[
\dot{\hat{x}}(t_i^-) = e^{A \tau_t} \hat{x}(t_{i-1}^-) + \int_0^{\tau_t} e^{A \sigma} B_c u(t_i - \sigma) d\sigma
\]

\[
\hat{Q}(t_i^-) = e^{A \tau_t} \hat{Q}(t_{i-1}^-) e^{A \tau_t^T} + W - e^{A \tau_t} W e^{A \tau_t^T}
\]

where \( W \) is the solution of the equation

\[
A_c W + W A_c^T = -W
\]

The equations (7), (8), (4), (5) and (6) define the algorithm of the Kalman filter for the general case of arbitrary measurement instants. This algorithm allows to estimate the states of the system at the measurement instants. In order to obtain the predicted value of the system output at the input updating periods \( (\hat{y}(kT)) \), it is necessary to obtain the state at that instants. This is accomplished by integrating the open loop equation (2) from the nearest past measurement instant \( t_j \), leading to

\[
\hat{y}(kT) = C_y \hat{x}(kT) = C_y \left\{ e^{A \tau_{kT}} \hat{x}(t_{j}^+) + \int_{kT-t_j}^{kT-t_{j+1}} e^{A \sigma} B_c u(kT - t_j - \sigma) d\sigma \right\}
\]

The main drawback of the general Kalman filter is the high computational complexity. On one hand there are several matrix exponentials to be calculated due to the arbitrary instants in which the measurements are taken. On the other hand the gain and covariance matrix equations are time varying and do not reach a stationary state due to the random availability of measurements (they have to be updated continuously). The computational complexity can be reduced if some restrictions are assumed on the measurement instants.

#### 3.2 Synchronous irregular sampling

If the measurements are taken at instants that are synchronous with the input update (even if they do not follow a periodic pattern), the equations (7) and (8) of the general algorithm can be greatly simplified. In that case all the measurement instants are multiple of the control rate \( T \). Defining \( k_i \in \mathcal{N}, \ i \in \mathcal{N} \) the set of integers such that \( k_i T \) are the measurement instants,
and \( N_i = k_i - k_{i-1} \) as the number of periods between measurements, the equations simplify to

\[
\hat{x}(k_i T^-) = A^{N_i} \hat{x}(k_{i-1} T^+) + \sum_{j=0}^{N_i-1} A^j B u((k_i - j - 1)T)
\]

\[
\dot{Q}(k_i T^-) = A^{N_i} \dot{Q}(k_{i-1} T^+) A^{N_i T} + \mathbb{W} - A^{N_i} \mathbb{W} A^{N_i T}
\]

where \( A = e^{A_i T} \) and \( B = \int_0^T e^{A_i \sigma} B \, d\sigma \). The equations (4), (5) and (6) are identical (changing \( t_i \) by \( k_i T \) and \( t_i - 1 \) by \( k_i - 1 T \)). In this case, the output equation (10) also simplifies to

\[
y(k T) = C_y \hat{x}(k T) = C_y \{ A^{k-k_j} \hat{x}(k_j T) + \sum_{p=0}^{k-k_j-1} A^p B u((k - k_j - p - 1)T)\}
\]

where \( k_j T \) is the nearest past measurement instant.

The resulting algorithm does not need to calculate exponential matrices. However, it still implies a high computational cost due to the time varying nature of the gain and matrix equations, that do not reach a stationary state if the measurements do not follow a periodic pattern.

### 3.3 Synchronous periodic sampling

If the synchronous measurement instants follow a periodic pattern, the equations of the Kalman filter can be simplified if the gain and matrix equations are assumed to reach a stationary state. This is only possible if the disturbances are also stationary.

The simplest case results from the assumption that the matrix that defines which elements of \( z(t) \) are measured is constant \((M_i = M)\) and the measurement instants are regular \((k_i = iN, \) where \( N \) is a constant value that represents the number of input periods between consecutive measurements). In this case the equations (12), (5) and (6) reach a stationary state whose solutions are constant \( \dot{Q} \) and \( L \) matrices. The stationary equations are

\[
\dot{Q}^- = A^{N} \dot{Q}^- A^{N T} + \mathbb{W} - A^{N} \mathbb{W} A^{N T}
\]

\[
L = \dot{Q}^- (M C_z)^T \left[ M C_z \dot{Q}^- (M C_z)^T + M V M^T \right]^{-1}
\]

\[
\dot{Q}^+ = \left[ I - L M C_z \right] \dot{Q}^-
\]

that can be joint into a discrete Ricatti equation

\[
\dot{Q} = A^{N} \left( \dot{Q} - \dot{Q} (M C_z)^T \left[ M C_z \dot{Q} (M C_z)^T + M V M^T \right]^{-1} M C_z \dot{Q} \right) A^{N T} + \mathbb{W} - A^{N} \mathbb{W} A^{N T}
\]

Solving this equation for matrix \( \dot{Q} \) and applying the equation (15) one obtains the constant matrix gain \( L \) that implements the stationary Kalman filter. The final state estimator equation is very simple in this case

\[
\hat{x}(i N T^+) = (I - L M C_z) \{ A^N \hat{x}((i - 1) N T^+) + \sum_{j=0}^{N-1} A^j B u((iN - j - 1)T) \} + L M z(i N T)
\]

The output at the control updating instants should then be calculated using the equation (13).

For arbitrary periodic measurement patterns the derivation of the stationary equations is more complicated. In order to do so, let us define the following matrices that univocally describe a periodic measurement pattern

1. \( \{ N_1 \cdots N_p \} \). Is a set of \( p \) scalar values that define the number of input periods between two consecutive measurement instants. Let us define \( N = \sum_{i=1}^{p} N_i \) as the global period of the measurement pattern.

2. \( \{ M_1, \ldots, M_p \} \). Is a set of \( p \) matrices (as defined in the previous section) that define which elements of vector \( z(t) \) are measured at every instant of the measurement pattern.

As an example, if \( m = 3 \) and the sampling pattern is defined such that \( \{ z_1, z_2, z_3 \} \) are measured at instant \( T \), \( \{ z_1 \} \) is measured at instant \( 3T \), \( \{ z_2 \} \) is measured at instant \( 4T \), \( \{ z_1, z_3 \} \) are measured at instant \( 5T \) and again \( \{ z_1, z_2, z_3 \} \) are measured at instant \( 7T \), the previous matrices are defined as \( N_1 = 2, N_2 = 1, N_3 = 1, N_4 = 2, M_1 = I; M_2 = [1 0 0]; M_3 = [0 1 0]; M_4 = [1 0 0 ; 0 0 1] \).

The stationary Kalman filter is now defined by \( p \) matrix gains \( L_1, \ldots, L_p \) (one for each sampling instant) and \( p \) variance-covariance matrices. The equations that define these matrices are obtained by applying the general equations at every sampling instant, leading to a set of \( 3p \) matrix equations

\[
\dot{Q}^-_j = A^{N_j} \dot{Q}^-_{j-1} A^{N_j T} + \mathbb{W} - A^{N_j} \mathbb{W} A^{N_j T}
\]

\[
L_j = \dot{Q}^-_j (M_j C_z)^T \left[ M_j C_z \dot{Q}^-_j (M_j C_z)^T + M_j V M_j^T \right]^{-1}
\]

\[
\dot{Q}^+_j = \left[ I - L_j M_j C_z \right] \dot{Q}^-_j
\]

where \( j = 1, \ldots, p \) and \( \dot{Q}_0 = \dot{Q}_p \). The resulting stationary Kalman filter is given by an equation like (18) where \( N, M \) and \( L \) are changed by \( N_j, M_j \) and \( L_j \) that rotate indefinitely from \( j = 1 \) till \( j = p \).

### 4 Application examples

#### 4.1 General case of asynchronous measurement

Consider the system shown in the figure 2. The input of the system is the force and it is assumed to be updated at a constant rate of \( T = 0.5s \). The measured signals are \( \theta \) and \( r \), but they are assumed to be measured by binary sensors that produce a digital pulse when the signal reaches some predefined values. A low cost encoder produces a pulse every 1 m for the \( r \) signal. The \( \theta \) signal is assumed to produce a pulse at values \( \{-0.07 \ -0.02 \ 0.02 \ 0.07\} \). As a consequence there are measurements of both \( \theta \) and \( r \) at random instants that are asynchronous with the input update. The linearized model of this system (see [2]) is defined
The resulting stationary Kalman filter is given by an equation like (18) where
\( N, M \) ... be \( \mu \), and hence \( C_y = [0 \ 0 \ 1 \ 0] \). The
objective of the predictor is then to estimate the value of \( \mu \) at the synchronous
instants (i.e. to obtain \( \tilde{\theta}(kT) \)). For the simulation, the input is generated as a band limited white
noise of period 1 sec. and power 100, that is contaminated by another band limited white noise of power 2 and period 0.2 sec. The measurements are contaminated by additive random normal noise of variance 0.01\(^2\)rad\(^2\) and 0.1\(^2\)m\(^2\) respectively. A Kalman filter is designed as described in section 3, taking first the
matrices \( V = [0.1^2 \ 0 \ 0 \ 0.01^2] \) and \( W = 4B_cB_c^T \). In the
figure 3 the true and the estimated states at the measurement instants are shown. In the figure 4 the true output \( \theta(kT) \) and the predicted one \( \tilde{\theta}(kT) \) at the input updating instants are shown. The average quadratic prediction error is \( 1.357 \cdot 10^{-5} \). If the \( V \) and \( W \) matrices are not exactly known, the behavior is not optimal, but the predictor still works well. As an example, if the matrices used in the predictor equation are \( V = [0.1^2 \ 0 \ 0 \ 0.02^2] \) and \( W = 2B_cB_c^T \), the quadratic prediction error average is \( 1.94 \cdot 10^{-5} \), and with \( V = [0.1^2 \ 0 \ 0 \ 0.005^2] \) and \( W = 8B_cB_c^T \), the quadratic prediction error average is \( 2.52 \cdot 10^{-5} \).

4.2 Synchronous output measurement with missing data

Let us consider the special case when the measured signal is the output to be controlled (i.e. \( C_z = C_y \)) and the measurement instants are synchronous with the input update. However, not all the measurements are assumed to be available (some or most of them are missing). The Kalman filter described in section 3.2 can be directly applied. However, in this special case other simpler predictors based on the input-output model can also be used. In [3] a very simple predictor based on the substitution in the difference equation of estimated outputs by the measurements when they are available is proposed. The difference equation used in the predictor is extended by multiplying the numerator and denominator of the discrete transfer function by a polynomial in order to reach an adequate dynamics. If the measurement pattern is regular, the error dynamics is defined by the eigenvalues of a matrix that depends on the sampling pattern and the process discrete model (see [3] for details).

In order to compare the two approaches, let us consider the system whose model is defined by

\[
A_c = \begin{bmatrix} -0.1985 & -0.0221 \\ 1 & 0 \end{bmatrix} ; \quad B_c = [1 \ 0]^T \quad C_z = [0 \ 0.254]
\]

An input updating period of \( T = 1s \) is assumed. A little error is assumed on the available model (-0.1985 is assumed to be -0.196 and -0.0221 is assumed -0.0243). The input to the system is a known sinusoidal signal of amplitude 1. An output disturbance of \( \sigma_z = 0.05 \) and a state disturbance of \( \sigma_x = 0.05 \) are assumed. The output is assumed to be available every \( N = 3 \) input periods. In this case a stationary Kalman filter is calculated based on the nominal model. The predictor based on the input output model uses a difference equation that consists of the discrete ZOH equivalent transfer function extended by the polynomial \( 1 + 0.9q^{-1} + 0.75q^{-2} \) (error eigenvalues \( 0.645 \pm 0.05j \)). In the figure 5 the prediction error is shown for both algorithms. Both predictors attain similar performances, but the behavior of the Kalman filter is slightly worse (i.e. it is not optimal) due to the slight modelling error.
4.3 Sensor fusion

Consider the case when the output of the system is measured by 2 sensors with different precision and measurement availability instants. This idea is covered by the general model (1) taking $C_z = [C_y^T C_y^T]^T$ and $V = [v_1 v_12 : v_{12} v_2]$. As an example consider the system described in section 4.1 and assume that only $\theta$ is measured with 2 sensors. The first one is a binary sensor fixed on position $\theta = 0$ that gives scarce but very precise measurements (null noise variance assumed). The second sensor is a continuous sensor that gives a measurement every $N = 2$ input updating periods but with a noise variance of $0.006^2$ (hence $V = [0.006^2 0 : 0 0]$. With the same input conditions as in section 4.1 the average quadratic prediction error is $4.35 \cdot 10^{-6}$. In order to compare the sensor fusion effect of the predictor, the same simulation is carried out assuming that only the second sensor is available, obtaining an error of $10.5 \cdot 10^{-6}$. If only the first sensor is used the resulting error is $10.7 \cdot 10^{-6}$. In the figure 6 the prediction errors are shown.

5 Conclusions

Virtual sensors are studied as systems that predict the output of the process at regular instants from irregularly sampled signals. A general predictor based on a Kalman filter that can be applied for arbitrary measurement conditions is described. The particularization of this predictor for different sampling situations is then derived. Some simulated examples demonstrate the validity of the approach and illustrate the possible applications. The first one considers a measurement device consisting of binary sensors that produce precise values at random instants. The second one assumes that the output is measured synchronously with the input update, but some of the measurements are missing. In this case the Kalman based predictor is compared to an input-output model based predictor showing a similar performance. The last example describes the sensor fusion where 2 sensor measure the same output with different precision and availability, showing how the output prediction is improved if both sensors are combined in the predictor algorithm.

References


