Product-Form Queuing Networks

- Consider a modular system consisting of multiple subsystems
- The more independent subsystems, the easier overall system analysis
  - For example, each subsystem can be modeled as a queue and the overall system analysis can be obtained from the product of terms corresponding to each queue!
  - We call such queuing networks product-form networks.
There are several properties that lead to product-form (PF) queuing networks:
- Local balance
- Reversibility
- Quasi-reversibility
- Station balance
- $M \Rightarrow M$
The effective rate at which the system leaves state S due to the service completion (of a chain r customer) at station i equals the effective rate at which the system enters state S due to an arrival (of chain r customers) to station i.
Reversible Markov Process

- Consider a film we make from the behavior of an isolated service station and then run the film backward.
  - The departing customers in the real system will be seen as arrival in backward run.
  - Let \( X(t) \) denote the random process describing the number of customers at the station.
  - Let \( X_b(t) \) denote the corresponding process in the backward run of the film.
  - If \( X(t) \) and \( X_b(t) \) are statistically identical, we say \( X(t) \) is reversible.
A random process $X(t)$ is reversible,

- If for any finite sequence of time instants $t_1, \ldots, t_k$ and a parameter $\tau$, the joint distribution of $X(t_1), \ldots, X(t_k)$ is the same as that of $X(\tau-t_1), \ldots, X(\tau-t_k)$
- $M/M/c$ queue is a reversible system
- $M/M/1$ queue with batch arrival is not reversible
Reversible Markov Process (Con.)

- A reversible process must be stationary
  - $X(\tau+t_1), \ldots, X(\tau+t_k)$ has the same distribution as $X(-t_1), \ldots, X(-t_k)$ which has the same distribution as $X(t_1), \ldots, X(t_k)$.
  - Thus all joint distribution are independent of time shift as required for stationary
Reversible Markov Process (Con.)

- Reversible Markov processes can be characterized by an important property known as **detailed balance**.

- **Lemma**: Let $X(t)$ be a stationary, discrete parameter Markov chain with one-step transition matrix $Q[q(i,j)]$. If $X(t)$ is reversible than it satisfy detailed balance equation:
  \[ P(i)q(i,j) = P(j) q(j,i) \]
Similarly, a continuous Markov chain is reversible if $\forall i,j. P(i)q_{i,j} = P(j)q_{i,j}$ where $q_{i,j}$ is the transition rates.

Conversely if we can find probabilities $P(i)$’s with $\Sigma P(i)=1$ satisfying detailed balance property, then $X(t)$ is reversible and $P(i)$’s form its stationary distribution.
Lemma (Kolmogorove criterion)

A stationary Markov chain is reversible if and only if for every finite sequence of states $i_0, \ldots, i_k$ the transition probability (or rates in the continuous parameter case) satisfy the following equation:

$$q(i_0, i_1) q(i_1, i_2) \cdots q(i_{k-1}, i_k) q(i_k, i_0) = q(i_k, i_{k-1}) q(i_{k-1}, i_{k-2}) \cdots q(i_1, i_0) q(i_0, i_k)$$
**Lemma**: Let $X(t)$ denote an ergodic Markov chain with transition matrix $Q$. Then $X(\tau-t)$ has the same stationary distribution as $X(t)$. Let $Q^*=[q^*(i,j)]$ denote the transition matrix of $X(\tau-t)$, then:

$$q^*(i,j)=P(j)q(j,i)/P(i)$$

- In other words, for a reversible process $Q=Q^*$
Example: Consider M/M/c queue

- Global balance results:
  \[ P(n)(\lambda + c\mu) = P(n-1)\lambda + P(n+1)c\mu \]

- Local balance results:
  \[ P(n)(\lambda) = P(n+1)c\mu \]

- Local balance is contained in global balance.
  - Thus \( P_{i}q_{i,j} = P_{j}q_{j,i} \) and is reversible.

Number of leaves has Poisson distribution with rate \( \lambda \).
Reversibility of M/M/c results in following lemma.

- The departure process of M/M/c system is Poisson and the number in the queue at any time $t$ is independent of the departure process prior to $t$.
- Thus we can easily analyze feed-forward network of M/M systems
Reversible Markov Process (Con.)

Example: Find $P(n_1, n_2)$

- $P(n_1) = (1-\rho_1)\rho^{n_1}$
- $P(n_2) = (1-\rho_2)\rho^{n_2}$

- The arrival to second queue (departure from first queue) is independent of number of customers at first queue (Previous Lemma).

- $P(n_1, n_2) = (1-\rho_1) (1-\rho_2) \rho^{n_1} \rho^{n_2}$
Reversible Markov Process (Con.)

Example

- $P(n_1, n_2, n_3) = (1-\rho_1)(1-\rho_2)(1-\rho_3)\rho^{n_1}\rho^{n_2}\rho^{n_3}$
- Avg. Queue Length = $ho_1/(1-\rho_1) + \rho_2/(1-\rho_2) + \rho_3/(1-\rho_3)$
- $W = \text{Avg. Queue Length} / \lambda$
- What if there is a feed-back from third queue to first queue: Quasi-reversibility preserves product-form property
Quasi-Reversible Queuing Systems

- Let $X(t)$ denote the queue length process at a queuing system. Then $X(t)$ is quasi-reversible if for any time instant $t_0$, $X(t_0)$ is independent of
  - Arrival time of customers after $t_0$
  - Departure times of customer prior to $t_0$
Open Single-Chain Product-Form Networks

- $q_{si}$: Probability that an external arrival is directed to station $i$ (similarly $q_{id}$ is defined)
- $\Lambda(n)$: Generation of customers at source with Poisson distribution.
- $\Lambda_i(n)$: External arrival to station $i$ ($q_{si}\Lambda(n) = \Lambda_i(n)$)
- $\mu_i$: basic service rate of station $i$
- $C_i(n)$: capacity of station $i$ at load $n$
Open Single-Chain Product-Form Networks (Con.)

- $v_i$: the visit ratio of station i relative to the source node
- $\lambda_i(n)$: the arrival rate of station i when there n customers in the entire network ($\lambda_i(n) = v_i \Lambda(n)$)
- Visit ratio to source and so destination should be 1
- By flow balance we have:

$$\sum_{i=1}^{M} q_{si} = 1 \quad \text{and} \quad \sum_{i=1}^{M} v_i q_{id} = 1$$

$$q_{id} = 1 - \sum_{j=1}^{M} q_{ij} \quad \text{and} \quad v_i = q_{si} + \sum_{j=1}^{M} v_j q_{ji} = 1$$
Open Single-Chain Product-Form Networks (Con.)

- $n=(n_1,\ldots,n_M)$: state of network where $n_i$ is the number of customers at station $i$.
- $e_i=(0,\ldots,0,1,0,\ldots,0)$: 1 at position $i$.
- The rate at which the system leaves state $n$ due to an arrival or completion at station $i$ is $q_{si}\Lambda(n)+\mu_i(n_i)q_{id}\sum\mu_i(n_i)q_{ij}$ or $q_{si}\Lambda(n)+\mu_i(n_i)$.
The global balance equation results

\[
P(n) \sum_{i=1}^{M} \Lambda(n)q_{si} + \mu_i(n_i) = \sum_{i=1}^{M} \sum_{j=1}^{M} \mu_j(n_j+1)P(n-e_i+e_j)q_{ji} + \sum_{i=1}^{M} \Lambda(n-1)P(n-e_i)q_{si} + \sum_{i=1}^{M} \mu_i(n_i+1)P(n+e_i)q_{id}
\]
Open Single-Chain Product-Form Networks (Con.)

- Since all stations are individually quasi-reversible, the entire network should be quasi-reversible:

\[ \mu_i(n_i)P(n) = \lambda_i(n-1)P(n-e_i), \; i=1,..,M \]

- We can find \( P(n) \) for station \( i \) and then the desired product-form solution for network:

\[ P(n) = \prod_{k=0}^{n-1} \Lambda(k) \prod_{i=1}^{M} \frac{u_i^{n_i}}{V_i(n_i)} P(0) \]