Homework 3 (Stochastic Processes)

1. Explain why each of the following is NOT a valid autocorrelation function:
   
   (a) \( R_X(\tau) = \begin{cases} e^{-\tau} & \tau \geq 0 \\
                               e^{2\tau} & \tau < 0 \end{cases} \)
   
   (b) \( R_X(\tau) = e^{-\tau} I_{|\tau| \leq 1} \)

   (c) \( R_X(\tau) = \frac{1+\tau^2}{1+\tau^4} \)

   All of them is considered for a continuous WSS process.

2. The discrete-time Linear system shown in the figure consists of one unit delay and a constant multiplier \((a < 1)\). The input to this system is a white noise with average power \(\sigma^2\). Find the spectral density and average power of the output.

3. Let \(\phi_1, \phi_2, \ldots, \phi_n\) be iid sample from the uniform distribution \(U(-\pi, \pi)\). Now consider a stochastic process \(X(t) = \sum_{i=1}^{n} a_i \sin \left(\frac{2\pi i}{n} t + \phi_i\right)\), Where \(a_i\) are constant coefficients.

   (a) Find \(R_X(t_1, t_2)\) ?

   (b) is \(X(t)\) a WSS process?

   (c) Consider an LTI system with impulse response \(h(t) = e^{-2t} u(t)\). If we apply \(X(t)\) as input to this system, Find \(R_{XY}\) and \(R_{YY}\).

4. Consider an LTI system with system function:

   \[ H(s) = \frac{1}{s^2 + 4s + 13} \]

   The input to this system is a WSS process \(X(t)\) with \(E\{X^2(t)\} = 10\). Find \(S_X(\omega)\) such that the average power of output is maximum.
5. Let \( \{X(t)\}, t \geq 0 \), be a second-order stationary process with covariance function \( C_X(\cdot) \). Set \( Y(t) = t^{-1} \int_0^t X(s)ds \). Show that

(a) \( Y(t) \) is a WSS process.
(b) \( C_Y(\tau) = 2\tau^{-2} \int_0^\tau (\tau - s)C_X(s)ds \)
(c) \( C_X(s) = \frac{1}{2s^2}[s^2C_Y(s)] \)

6. Let \( \{X(t)\}, t \in \mathbb{R} \), be a continuous wide-sense-stationary process with unknown mean \( m \) and covariance function \( Cov(s) = ae^{b|s|}, t \in \mathbb{R} \), where \( a > 0 \) and \( b > 0 \). For fixed \( T > 0 \) set \( \bar{X} = T^{-1} \int_0^T X(s)ds \). Show that

(a) \( E[\bar{X}] = m \). That is, \( \bar{X} \) is an unbiased estimator of \( m \).
(b) \( \text{var}(\bar{X}) = 2a[(bT)^{-1} - (bT)^{-2}(1 - e^{-bT})] \)

7. Let \( X_n = (X + na \mod 1), n = 0, \pm 1, \pm 2, \ldots \), where \( X \) is an RV uniformly distributed on \([0, 1]\) and \( a \) is a fixed irrational number. Show that \( X_n \) is strictly stationary.

8. Consider the random process \( X(t) \) as a stock index value at time \( t \). Then \( S(t) = X(t) - X(t-5) \) is representative of the return if someone buys the stock at time \( t-5 \) and sells it at time \( t \). Suppose \( S(t) \) is a normal WSS process with mean 0 and autocorrelation function \( R(\tau) = 5e^{-|\tau|} \). We have observed that \( S(6), S(7) \) are below 3. That is we know there is negative return if some buys at time 6 and sells at time 7, respectively. We want to estimate what is the probability that \( S(10) \) is positive. In other terms we want to see whether we have any chance to get profit if we buy the stock at time 10 and sell it at time 10.

9. Suppose \( V \) is a uniform random variable with \( U(0, 1) \) as its PDF. Define two stochastic processes \( X(t) = u(t-V) \) and \( Y(t) = \delta(t-V) \). (Assume \( 0 \leq t \leq 1 \)) a. Explain intuitively what \( X(t) \) and \( Y(t) \) describe. b. Evaluate the mean of \( X(t) \) and \( Y(t) \). (Assume \( 0 \leq t \leq 1 \)) c. Evaluate \( R_{XX}(t_1,t_2) \), \( R_{XY}(t_1,t_2) \) and \( R_{YY}(t_1,t_2) \). (Assume \( 0 \leq t_1 \leq 1 \) and \( 0 \leq t_2 \leq 1 \))

10. Show that if the processes \( x(t) \) and \( y(t) \) are WSS and \( E\{|x(0)-y(0)|^2\} = 0 \), then \( R_{xx}(\tau) \equiv R_{xy}(\tau) \equiv R_{yy}(\tau) \). Hint: Set \( z = x(t+\tau), w = x^*(t) - y^*(t) \) and use Schwartz’s inequality.