Homework 4 (Estimation theory)

1. \( T_1 \) and \( T_2 \) are equivalent statistics if, and only if, we can write \( T_2 = H(T_1) \) for some 1-1 transformation \( H \) of the range of \( T_1 \) into the range of \( T_2 \). Which of the following statistics are equivalent?

(a) \( \prod_{i=1}^{n} x_i \) AND \( \sum_{i=1}^{n} \log x_i, x_i > 0 \)
(b) \( \sum_{i=1}^{n} x_i \) AND \( \sum_{i=1}^{n} \log x_i; x_i > 0 \)
(c) \( (\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^2) \) AND \( (\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} (x_i - \bar{x}_i)^2) \)
(d) \( (\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^2) \) AND \( (\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} (x_i - \bar{x}_i)^3) \)

2. Suppose \( X_1, X_2, \ldots, X_n \) is a sample from following distributions. Find a Sufficient Statistic for each part.

(a) \( p(x|\theta) = \theta x^{\theta-1}, 0 < x < 1, \theta > 0 \)
(b) \( p(x|\theta) = \theta ax^{a-1}e^{-\theta x^a}, x > 0, \theta > 0, a > 0 \)
(c) \( p(x|\theta) = \theta e^{\theta x} \) AND \( \theta > 0, a > 0 \)

3. For each of the following distributions let \( X_1, \ldots, X_n \) be a random sample. Find a minimal sufficient statistic for \( \theta \).

(a) \( f(x|\theta) = \frac{1}{\sqrt{2\pi}} e^{-(x-\theta)^2/2}, \) \(-\infty < x < \infty, -\infty < \theta < \infty \)
(b) \( f(x|\theta) = e^{-(x-\theta)}, \) \( \theta < x < \infty, -\infty < \theta < \infty \)
(c) \( f(x|\theta) = \frac{e^{-(x-\theta)}}{(1+e^{-(x-\theta)})^2}, \) \(-\infty < x < \infty, -\infty < \theta < \infty \)
(d) \( f(x|\theta) = \frac{1}{\pi(1+(x-\theta)^2)}, \) \(-\infty < x < \infty, -\infty < \theta < \infty \)
(e) \( f(x|\theta) = \frac{1}{2} e^{-(x-\theta)}, \) \(-\infty < x < \infty, -\infty < \theta < \infty \)

4. Establish the following statements:

(a) The statistic \( (\sum X_i, \sum X_i^2) \) is sufficient, but not minimal sufficient, in the \( N(\mu, \mu) \) family.
(b) The statistic \( \sum X_i^2 \) is sufficient in the \( N(\mu, \mu) \) family.
(c) The statistic \( (\sum X_i, \sum X_i^2) \) is minimal sufficient, in the \( N(\mu, \mu^2) \) family.
(d) The statistic \( (\sum X_i, \sum X_i^2) \) is minimal sufficient, in the \( N(\mu, \sigma^2) \) family.

5. For each of the following pdfs let \( X_1, \ldots, X_n \) be iid observations. Find a complete sufficient statistic.

(a) \( f(x|\theta) = \frac{2x}{\theta}, 0 < x < \theta, \theta > 0 \).
(b) \[ f(x|\theta) = \frac{\theta}{(1+x)^{\theta+1}}, \quad 0 < x < \infty, \quad \theta > 0. \]

(c) \[ f(x|\theta) = \frac{(\log \theta)^x}{\theta^{x+1}}, \quad 0 < x < 1, \quad \theta > 1. \]

(d) \[ f(x|\theta) = \left(\frac{2}{\theta}\right)^x (1-\theta)^{2-x}, \quad x = 0, 1, 2, \quad 0 \leq \theta \leq 1. \]

6. Consider a person starting his random walk from point \( k \) on the X axis at time 0 \((0 \leq k \leq n)\). At each step, he moves one unit to the right or left, with same probability 0.5. He continues his walk, until he reaches one of points 0 or \( n \). If \( T \) is the R.V. which represents the time in which he reaches one of points 0 or \( n \), find \( E[T]\).

7. Consider \( n+1 \) points 0, 1, \ldots, \( n \) on a circle (point \( i \) is adjacent to points \((i+1) \mod (n+1)\) and \((i-1) \mod (n+1)\)). A person starts his random walk from point 0. At each step he moves to one of the adjacent points with same probability 0.5. He continues his walk until he reaches point \( k \). If \( T \) is the R.V. which represents the time in which he reaches point \( k \), find \( E[T]\).

8. Consider a complete graph with \( n \) vertices (a graph in which there is an edge between each pair of vertices). A person starts his random walk from vertex 1. At each step, he moves to one of the adjacent vertices with same probability \( \frac{1}{n-1} \). If \( S \) is the R.V. which represents the time it takes to visit all vertices, find \( E[S]\).

9. Consider a person who randomly walks on the vertices of a graph \( G \), starting from a node \( v \). At each step, he moves to one of the adjacent vertices with the same probability. He continues until he reaches vertex \( u \). Let \( T_{vu} \) be the R.V representing the time in which he reaches \( u \). Give an example in which adding an edge to \( G \) which reduces the length of shortest path between \( v \) and \( u \), causes increase in \( T_{vu} \).